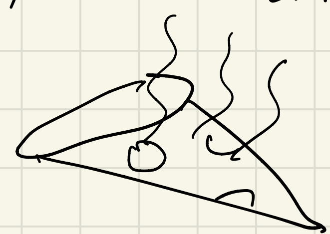
A $\boxed{\text{graph}}$ G is defined by a pair of sets,

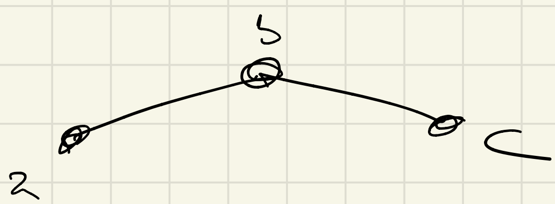
$V(G) :=$ elements called its $\boxed{\text{vertices}}$

$E(G) :=$ 2-element subsets of $V(G)$.

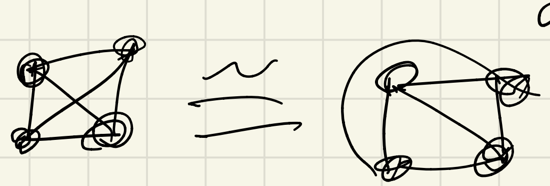
A graph is $\boxed{\text{connected}}$ if for every pair of vertices, u & v, there exists a sequence of edges $\{x_i, y_i\}$ w/ $y_i = x_{i+1}$ & u, v both somewhere



eg: G: $V(G) = \{a, b, c\}$; $E(G) = \{\{a,b\}, \{b,c\}\}$

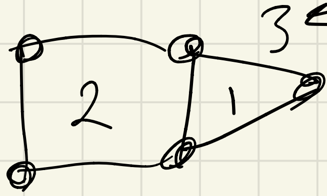


A graph is $\boxed{\text{planar}}$ if it can be drawn with no edges crossing.

eg: 

The regions described by edges in a drawing of a planar graph are called $\boxed{\text{faces}}$

eg:



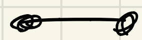
3 ← outer face

3 faces total.

Thm 1: (Euler's Formula) G planar, connected, $|E| \geq 1$

$$\Longrightarrow \underline{|V| + |F| = |E| + 2}$$

(Pf) (BC) $|E| = 1$



$2+1 = 1+2$ ✓

(IS) Spoz true for $|E|$, prove for $|E|+1$

add a face

add a vertex

$$|V| + |F| = |E| + 2$$

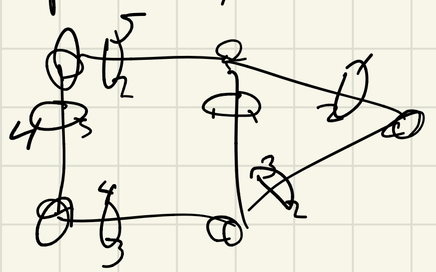
$$+1 \quad +1$$

✓

∎

Thm 2: G planar, connected $|E| \geq 2 \Rightarrow 3|F| \leq 2|E|$

(pf)



$$3|F| \leq \#\text{dog ears} = 2|E| \quad \blacksquare$$

Cor 3: G planar, connected $\Rightarrow |E| - 3|V| + 6 \leq 0$

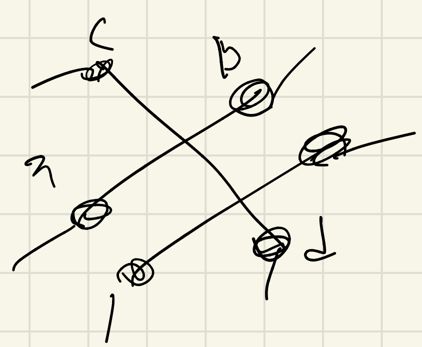
(pf) Thm 1 $\Rightarrow |F| = \underbrace{|E| + 2 - |V|}$

Thm 2 $\Rightarrow 3(|E| + 2 - |V|) \leq 2|E| \quad \blacksquare$

The $\boxed{\text{crossing number}}$ of a graph G $cr(G)$ is the minimum number of edge crossings under any redrawing.

Cor 4: $cr(G) \geq \underline{|E| - 3|V| + 6}$

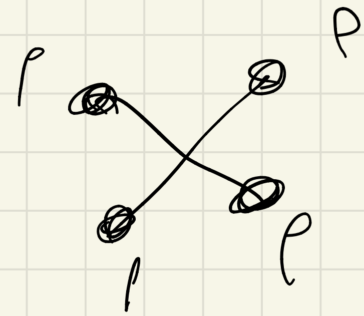
(pf)



removing $\{a, b\}$ loses ≥ 1 crossing

removing $\{c, d\}$ loses ≥ 2 crossings... $\blacksquare$

Given a graph G, define a random induced (keep edges) subgraph H by keeping each vertex with probability $p \in (0,1)$



$$\mathbb{E}(V(H)) = |V(G)|\,p$$

$$\mathbb{E}(E(H)) = |E(G)|\,p^2$$

$$\mathbb{E}(cr(H)) \leq cr(G)\,p^4$$

Recall Cor 4 $\Rightarrow$

$$cr(G) \geq |E(G)| - 3|V(G)| + 6$$

$$\mathbb{E}(cr(H)) \geq \mathbb{E}(|E(H)|) - \mathbb{E}(3|V(H)|) + \mathbb{E}(6)$$

$$\geq$$

$$p^4\, cr(G) \qquad = p^2|E(G)| - 3p|V(G)| + 6$$

$$cr(G) \geq \frac{p^2|E(G)| - 3p|V(G)| + 6}{p^4}$$

$$cr(G) \geq \frac{p^2|E(G)| - 3p|V(G)| + 6}{p^4}$$

What if $p^2|E(G)| > 3p|V(G)|$

$$\Rightarrow p > 3|V(G)| / |E(G)|$$

Let's set $p = 4|V(G)| / |E(G)|$

$$\Rightarrow cr(G) \geq \frac{\left(\frac{4V}{E}\right)^2 \cdot E - 3\left(\frac{4V}{E}\right) \cdot V}{\left(\frac{4V}{E}\right)^4}$$

$$\Rightarrow cr(G) \geq \frac{16V^2E^{-1} - 12V^2E^{-1}}{256V^4E^{-4}}$$

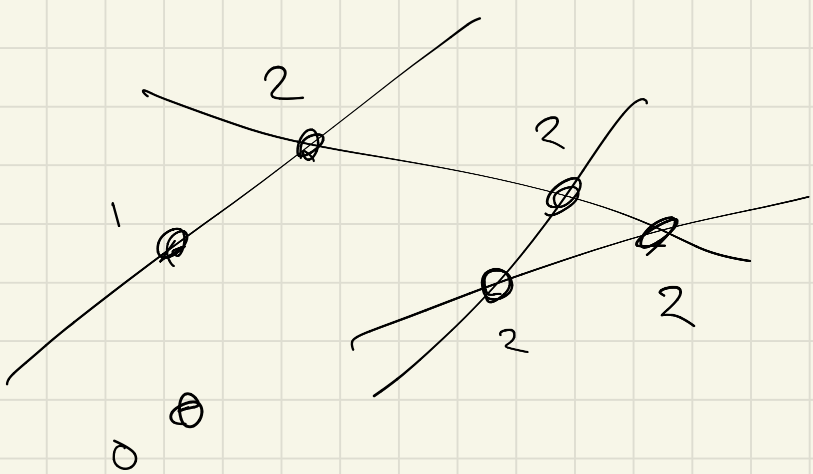
$$\Rightarrow cr(G) \geq \frac{4V^2E^4}{E \cdot 256V^4} = \frac{E^3}{64V^2}$$

CROSSING NUMBER LEMMA:

<u>Thm</u>: $|E(G)| > 4|V(G)| \Rightarrow$

$$cr(G) \gtrsim |E(G)|^3 / |V(G)|^2$$

Given a set of points P & lines $\mathcal{L}$, an $\boxed{\text{incidence}}$ is a pair $(p, \ell) \in P \times \mathcal{L}$, where $p \in \ell$.



$\Rightarrow$ 9 incidences

$I(P, \mathcal{L}) :=$ # of incidences

between points in P & lines in $\mathcal{L}$

Thm (Szemerédi-Trotter): for any set of n points, P, & m lines, $\mathcal{L}$ in $\mathbb{R}^2$,

$$I(P,\mathcal{L}) \lesssim (nm)^{2/3} + n + m$$

(pf) Let G be a graph whose vertices are P & whose edges are segments between consecutive points on a line from $\mathcal{L}$. $I(P,\{\ell\}) = \#\text{edges on } \ell + 1$.

$$\Rightarrow I(P,\mathcal{L}) = |E(G)| + m$$

Case 1: $|E(G)| \leq 4|V(G)|$

$$\Rightarrow I(P,\mathcal{L}) - m \leq 4n \Rightarrow I \lesssim n+m$$

Case 2: $|E(G)| > 4|V(G)|$

$$\frac{E^3}{V^2} \lesssim cr(G) \lesssim m^2$$

(total possible crossings from m lines)

$$(I-m)^3 \lesssim m^2 n^2$$

$$\Rightarrow I \lesssim (nm)^{2/3} + m \qquad \blacksquare$$