(1)

$$F(R) = \int_a^b e^{iRf(x)}\,dx =$$

$$\int_a^b \left(e^{iRf(x)}\right)' \frac{1}{iRf'(x)}\,dx$$

$$= \underbrace{e^{iRf(x)} \frac{1}{iRf'(x)}\Big|_a^b}_{\overline{\overline{I}}} - \underbrace{\frac{1}{iR}\int_a^b e^{iRf(x)} \left(\frac{1}{f'(x)}\right)' dx}_{\overline{\overline{II}}}$$

$|I| \le \frac{2}{R}$ if we assume that

$f \in C'(a,b) \quad f'(x) \ge 1$

$$\left| \frac{1}{iR}\int_a^b e^{iRf(x)} \left(\frac{1}{f'(x)}\right)' dx \right|$$

$$\le \frac{1}{R}\int_a^b \left| \left(\frac{1}{f'(x)}\right)' \right| dx$$

If f' is monotonic, the previous quantity

$$= \frac{1}{R}\left|\int_a^b \left(\frac{1}{f'(x)}\right)' dx\right| =$$

$$\frac{1}{R}\left|\frac{1}{f'(b)} - \frac{1}{f'(a)}\right| \leq \frac{2}{R}.$$

Lemma: $f \in C^1(a,b)$, f' monotonic

Then $\left|\int_a^b e^{iRf(x)}\, dx\right| \leq \frac{4}{R}$.

Bad news: $a=0$ $b=1$ $f(x)=x^2$

$f'(0)=0$

$$\int_{-1}^{1} e^{iRx^2}\,dx \qquad u = x\sqrt{R}$$

$$du = \sqrt{R}\,dx$$

$$\frac{1}{\sqrt{R}}\int_{-\sqrt{R}}^{\sqrt{R}} e^{iu^2}\,du$$

Exercise: prove that $\left|\int_{-\sqrt{R}}^{\sqrt{R}} e^{iu^2}\,du\right| \le C$

independent of R

$f(x) = x^2 \qquad f''(x) = 2 \qquad f'(c) = 0$

$$\int_a^b e^{iRf(x)}\,dx = \underbrace{\int_a^{c-\delta}\cdots}_{\mathrm{I}} + \underbrace{\int_{c-\delta}^{c+\delta}\cdots}_{\mathrm{II}} + \underbrace{\int_{c+\delta}^{b}\cdots}_{\mathrm{III}}$$

(4)

$$\int_a^b e^{iRf(x)}\,dx = \underbrace{\int_a^{c-\delta} \cdots}_{I} + \underbrace{\int_{c-\delta}^{c+\delta} \cdots}_{II} + \underbrace{\int_{c+\delta}^{b} \cdots}_{III}$$

$$|II| = \left| \int_{c-\delta}^{c+\delta} e^{iRf(x)}\,dx \right| \leq \int_{c-\delta}^{c+\delta} dx = 2\delta$$

$$\underbrace{f'(x) - f'(c)}_{(c+\delta,\, b)} = (x-c)\,f''(\text{something}) \geq \delta$$

$$\int_{c+\delta}^{b} e^{iRf(x)}\,dx = \int_{c+\delta}^{b} e^{\underbrace{iR\delta}\left(\frac{f(x)}{\delta}\right)}\,dx$$

$$\left(\frac{f(x)}{\delta}\right)' = \frac{f'(x)}{\delta} \geq \frac{\delta}{\delta} = 1$$

$$|III| \leq \frac{4}{R\delta}$$

(5)

$$F(R)=\int_a^b e^{iRf(x)}\,dx = \overline{\underline{I}} + \overline{\underline{II}} + \overline{\underline{III}}$$

$$|\overline{\underline{I}}|, |\overline{\underline{III}}| \leq \frac{4}{R\delta} \qquad |\overline{\underline{II}}| \leq 2\delta$$

$$|F(R)| \leq \frac{8}{R\delta} + 2\delta$$

Choose $\delta = \frac{1}{\sqrt{R}}$

$$|F(R)| \leq \frac{8}{\sqrt{R}} + \frac{2}{\sqrt{R}} = \frac{10}{\sqrt{R}}$$

Back of the envelope calculation:

$$\frac{1}{R\delta} \sim \delta \qquad \frac{1}{R} \sim \delta^2 \qquad \delta \sim \frac{1}{\sqrt{R}}$$

Lemma: $f \in C^2(a,b)$, $f''(x) \geq 1$

Then $\left| \int_a^b e^{iRf(x)}\,dx \right| \leq \frac{10}{\sqrt{R}}$

(6)

$$B_d = \{ x \in \mathbb{R}^d : |x| \leq 1 \}$$

$$\int_{\mathbb{R}^d} e^{-|x|^2} dx$$

$$\parallel$$

$$\int \dots \int e^{-x_1^2} \dots e^{-x_d^2} dx_1 \dots dx_d = \left(\int e^{-t^2} dt \right)^d = \pi^{\frac{d}{2}}$$

$$\int_{\mathbb{R}^d} e^{-|x|^2} dx = \int_0^\infty \int_{S^{d-1}} e^{-r^2} r^{d-1} d\omega \, dr$$

$$= |S^{d-1}| \int_0^\infty e^{-r^2} r^{d-1} dr \qquad u = r^2 \quad du = 2r\,dr$$

$$= |S^{d-1}| \int_0^\infty e^{-u} u^{\frac{d}{2}-1} \frac{1}{2} du = |S^{d-1}| \, \Gamma\left(\frac{d}{2}\right)$$

$$\Gamma(\lambda) = \int_0^\infty e^{-u} u^{\lambda - 1} du$$

⑦

$$\pi^{\frac{d}{2}} = \int_{\mathbb{R}^d} e^{-|x|^2} dx = \tfrac{1}{2} |S^{d-1}| \, \Gamma\left(\tfrac{d}{2}\right)$$

$$\Gamma(\lambda) = \int_0^\infty e^{-u} u^{\lambda-1} du$$

Claim: $\Gamma(n) = (n-1)!$

$$|S^{d-1}| = 2\pi^{\frac{d}{2}} / \Gamma\left(\tfrac{d}{2}\right)$$

$$\int_{|x|\le 1} dx = \int_0^1 \int_{S^{d-1}} r^{d-1} \, d\omega \, dr = \frac{|S^{d-1}|}{d}$$

$$Vol(B_d) = 2\pi^{\frac{d}{2}} / d \, \Gamma\left(\tfrac{d}{2}\right)$$

$d = 2n$

$$\text{volume} = \frac{2\pi^n}{2n\,\Gamma(n)} = \frac{\pi^n}{n!} \xrightarrow[n\to\infty]{} 0$$

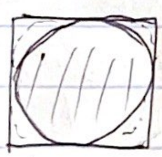
$\Gamma(n) = (n-1)!$

$$\Gamma(n) = \int_0^\infty e^{-u} u^{n-1}\,du = \int_0^\infty -\left(e^{-u}\right)' u^{n-1}\,du$$

$$= \underbrace{e^{-u} u^{n-1}\Big|_0^\infty}_{=0} + \underbrace{\int_0^\infty e^{-u} (n-1) u^{n-1-1}\,du}_{=(n-1)\Gamma(n-1)}$$

$\{a_n\}$ $\quad a_1 = 1 \quad a_n = (n-1)\,a_{n-1} \quad n \geq 2$

$a_n = (n-1)!$



$B_m = \{ x \in \mathbb{R}^d : x_1^m + x_2^m + \dots + x_d^m \leq 1 \}$

m even integen