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More Chapter 2

Chernoff's inequality: Let X_i be independent Bernoulli random variables w/ parameters p_i. Let

$S_N = \sum_i X_i$ and denote its mean by $\mu = \mathbb{E} S_N$.

Then for any $t > \mu$, we have

$$\mathbb{P}\{S_N \geq t\} \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^t$$

Proof: Just as in the proof of Hoeffding's inequality,

$$\mathbb{P}\{S_N \geq t\} \leq e^{-\lambda t} \prod_{i=1}^{N} \mathbb{E}\exp(\lambda X_i)$$

Now,

$$\mathbb{E}\exp(\lambda X_i) = e^{\lambda} p_i + (1-p_i)e^{0}$$

$$\leq \exp((e^{\lambda}-1)p_i) \quad \text{since } 1+x \leq e^{x}$$

Taylor expansion

(2)

It follows that

$$\prod_{i=1}^{N} \mathbb{E}\exp(\lambda X_i) \le \exp\left((e^{\lambda}-1)\sum_{i=1}^{N} p_i\right) =$$

$$\exp((e^{\lambda}-1)\mu)$$

It follows that

$$\mathbb{P}\{S_N \ge t\} \le e^{-\lambda t} \exp((e^{\lambda}-1)\mu)$$

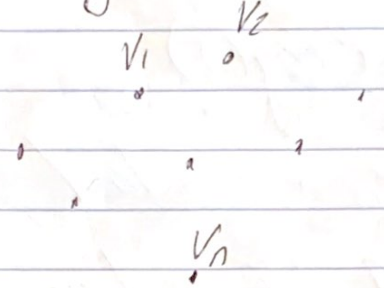
Let $\lambda = \ln(t/\mu)$

Then $e^{-\lambda t} \exp((e^{\lambda}-1)\mu)$

$$= \left(\frac{t}{\mu}\right)^{-t} \exp\left(\left(\frac{t}{\mu}-1\right)\mu\right) =$$

$\left(\frac{\mu}{t}\right)^{t} \exp(t-\mu)$ and we are done!

Erdös-Renyi model:



n vertices

$G(n,p)$

a given pair of vertices is connected by an edge w/ probability p (independently)

The degree of a vertex is defined as the number of edges incident to that vertex

The expected degree of every vertex is

$$(n-1)p \equiv \underline{\underline{d}}$$

We shall see that if $d \gtrsim \log(n)$, all the vertices have <u>essentially</u> the same number of vertices.

the whole substance here is buried in the meaning of the word "essentially"

Proposition: Consider a random graph $G \sim G(n,p)$ w/ expected degree $d \geq \underbrace{C}_{\text{uniform constant}}\log(n)$

Then with probability .9 all vertices of G have degrees between $.9d$ and $1.1d$.

Proof: Fix a vertex i of the graph.

The degree of i, denoted by d_i, is a sum of $n-1$ independent random variables $\mathrm{Ber}(p)$

By Chernoff, $\underbrace{\mathbb{P}\{|d-d_i| \geq .1d\}}_{} \leq 2e^{-cd}$

note that this is not exactly Chernoff; it is a modification given in the Exercise 2.3.5

The bound holds for each fixed i. The idea is to take a union:

$$\mathbb{P}\{\exists i \leq n : |d_i - d| \geq .1d\} \leq \sum_{i=1}^{n} \mathbb{P}\{|d_i - d| \geq .1d\}$$

$\leq n 2 e^{-cd} \leq .1$ if $d \geq C \log(n)$ w/ C large enough.

It follows that

$$\mathbb{P}\{\forall i \leq n : |d_i - d| < .1 d\} \geq .9$$

and we are done!

Sub-Gaussians: Which random variables X_i obey $\mathbb{P}\left\{\left|\sum_{i=1}^{N} a_i X_i\right| \geq t\right\} \underset{(*)}{\leq} 2\exp\left(-\frac{ct^2}{\|a\|_2^2}\right)$

Suppose that $\sum_{i=1}^{N} a_i X_i = X_i$ single term w/ coefficient 1.

Then $(*)$ takes the form

$$\mathbb{P}\{|X_i| \geq t\} \leq 2e^{-ct^2}$$

So if $(*)$ is to hold, random variables X_i must have sub-Gaussian tails.

The random variables w/ Sub-Gaussian tails are worth studying systematically. We shall take a small bite here.

It follows from Proposition 2.1.2 that if $X \sim N(0,1)$,

$$\mathbb{P}\{|X| \geq t\} \leq 2e^{-\frac{t^2}{2}} \quad \forall t \geq 0.$$

Let us now try to get a more general picture.

Proposition: X random variable. Then the following are equivalent.

i) The tails of X satisfy

$$\mathbb{P}\{|X| \geq t\} \leq 2\exp(-t^2/K_1^2) \quad \forall t \geq 0$$

ii) The moments of X satisfy

$$\|X\|_{L^p} = (\mathbb{E}|X|^p)^{\frac{1}{p}} \leq K_2\sqrt{p} \quad \forall p \geq 1.$$

iii) The MGF of X^2 satisfies

$$\mathbb{E}\exp(\lambda^2 X^2) \leq \exp(K_3^2\lambda^2) \quad \forall \lambda \text{ w/ } |\lambda| \leq \frac{1}{K_3}$$

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iv) The MGF of X^2 is bounded at some point, namely $\mathbb{E}\exp(X^2/K_4^2) \le 2$.

Moreover, if $\mathbb{E}X=0$, then i)-iv) are also equivalent to

v) The MGF of X satisfies

$$\mathbb{E}\exp(\lambda X) \le \exp(K_5^2\lambda^2)$$

$$\forall \lambda \in \mathbb{R}$$

Proof: i) $\hookrightarrow$ ii)

Assume that i) holds. We may assume that $K_1=1$ by replacing X w/ X/K_1.

We have

$$\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}\{|X|^p \ge u\}\,du$$

$$= \int_0^\infty \mathbb{P}\{|X| \ge t\}\,pt^{p-1}\,dt \quad (u=t^p)$$

$$\le \int_0^\infty 2e^{-t^2}pt^{p-1}\,dt = p\,\Gamma\left(\tfrac{p}{2}\right) \underset{\text{why?}}{\le} p\cdot\left(\tfrac{p}{2}\right)^{\frac{p}{2}}$$

The last step requires some elaboration. We need the fact that

$\Gamma(x) \le x^x$, which follows from Stirling's approximation for the gamma function:

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z \left(1 + O\left(\frac{1}{z}\right)\right)$$

and recall that $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx$

Taking p'th root yields ii)

We now prove that ii) $\Rightarrow$ iii)

Assume that property ii) holds. Once again, we may assume that $K_2 = 1$.

$$\mathbb{E} \exp(\lambda^2 X^2) \underset{\text{Taylor expansion}}{=} \mathbb{E}\left(1 + \sum_{p=1}^{\infty} \frac{(\lambda^2 X^2)^p}{p!}\right)$$

$$= 1 + \sum_{p=1}^{\infty} \frac{\lambda^{2p} \mathbb{E}(X^{2p})}{p!}.$$

(9)

By ii), $\mathbb{E}(X^{2p}) \leq (2p)^p$ ✓

Claim: $p! \geq \left(\frac{p}{e}\right)^p$

Then $\mathbb{E}\exp(\lambda^2 X^2) \leq 1 + \sum_{p=1}^{\infty} \frac{(2\lambda^2 p)^p}{(p/e)^p}$

$$= \sum_{p=0}^{\infty} (2e\lambda^2)^p = \frac{1}{1-2e\lambda^2}$$

if $2e\lambda^2 < 1$ ✓

Claim: $\frac{1}{1-x} \leq e^{2x}$ for $x \in [0, \frac{1}{2}]$

It follows that

$\mathbb{E}\exp(\lambda^2 X^2) \leq \exp(4e\lambda^2)$

$\forall \lambda$ w/ $|\lambda| \leq \frac{1}{2\sqrt{e}}$ $\hookrightarrow$ iii)

w/ $K_3 = \frac{1}{2\sqrt{e}}$

provided that we can establish the claims above.

The second claim can be proved as follows.

If $|x| < \frac{1}{2}$, $\dfrac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots$

We must show that

$$1 + x + x^2 + \ldots + x^n + \ldots \leq e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots + \frac{(2x)^n}{n!} + \ldots$$

The trick is to notice that it is enough to prove that

$$1 + x + x^2 + \ldots + x^n + \ldots \leq 1 + 2x \quad \text{if} \quad 0 < x < \frac{1}{2}$$

This reduces to

$$x^2 + \ldots + x^n + \ldots \leq x$$

$$\| $$

$$x^2(1 + x + x^2 + \ldots) \leq x$$

$$\| $$

$$\frac{x^2}{1-x} \leq x, \quad \frac{x}{1-x} \leq 1, \quad x \leq 1 - x \quad \checkmark$$

iii) $\hookrightarrow$ iv) is immediate ✓

iv) $\hookrightarrow$ i) Assume that iv) holds and reduce, as usual, to the case $K_4 = 1$.

Then $\mathbb{P}\{|X| \geq t\} = \mathbb{P}\{e^{|X|^2} \geq e^{t^2}\}$

$\leq e^{-t^2} \mathbb{E} e^{|X|^2}$ (Markov)

$\leq 2e^{-t^2}$ by property iv)

This proves property i) w/ $K_1 = 1$.