①

$1<p<\infty \qquad a_i, b_i \in \mathbb{R}$

$$\left(\sum_i |a_i+b_i|^p\right)^{\frac{1}{p}} \le \left(\sum_i |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_i |b_i|^p\right)^{\frac{1}{p}}$$

$$\|$$

$$\sum_i |a_i+b_i|^{p-1}|a_i+b_i| \le$$

$$\underbrace{\sum_i |a_i+b_i|^{p-1}|a_i|}_{\mathrm{I}} + \underbrace{\sum_i |a_i+b_i|^{p-1}|b_i|}_{\mathrm{II}}$$

$$\frac{1}{p}+\frac{1}{q}=1 \qquad \frac{1}{q}=1-\frac{1}{p}=\frac{p-1}{p}$$

$$q=\frac{p}{p-1}$$

$$\mathrm{I} \le \left(\sum_i |a_i|^p\right)^{\frac{1}{p}}\left(\sum_i |a_i+b_i|^{\cancel{(p-1)}\frac{p}{\cancel{p-1}}}\right)^{\frac{p-1}{p}}$$

$$\mathrm{II} \le \left(\sum_i |b_i|^p\right)^{\frac{1}{p}}\left(\sum_i |a_i+b_i|^p\right)^{\frac{p-1}{p}}$$

(2)

$$\sum_i |a_i+b_i|^p \le \left(\sum_i |a_i+b_i|^p\right)^{\frac{p-1}{p}} \left(\left(\sum_i |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_i |b_i|^p\right)^{\frac{1}{p}}\right)$$

$$\left(\sum_i |a_i+b_i|^p\right)^{\frac{1}{p}} \le \left(\sum_i |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_i |b_i|^p\right)^{\frac{1}{p}}$$

Arithmetic-Geometric Inequality:

$a_1, a_2, \dots, a_n \geq 0$ real numbers

Then $\underbrace{(a_1 a_2 \dots a_n)^{\frac{1}{n}}}_{\text{geometric mean}} \le \underbrace{\frac{a_1+a_2+\dots+a_n}{n}}_{\text{arithmetic mean}}$

$n=2 \quad (a_1 a_2)^{\frac{1}{2}} \le \frac{a_1+a_2}{2}$

$0 \le \left(a_1^{\frac{1}{2}} - a_2^{\frac{1}{2}}\right)^2 = a_1 + a_2 - 2a_1^{\frac{1}{2}} a_2^{\frac{1}{2}}$

(3)

$n=4$

$$(a_1 a_2 a_3 a_4)^{\frac{1}{4}} \leq \frac{a_1+a_2+a_3+a_4}{4}$$

$$\left(\left(a_1^{\frac{1}{2}} a_2^{\frac{1}{2}}\right) \cdot \left(a_3^{\frac{1}{2}} a_4^{\frac{1}{2}}\right)\right)^{\frac{1}{2}}$$

$$\leq \frac{a_1^{\frac{1}{2}} a_2^{\frac{1}{2}} + a_3^{\frac{1}{2}} a_4^{\frac{1}{2}}}{2} \leq \frac{\frac{a_1+a_2}{2} + \frac{a_3+a_4}{2}}{2}$$

$$= \frac{a_1+a_2+a_3+a_4}{4}$$

A slight elaboration on the above yields the case $n=2^k$.

Now we must fill in all the $n$'s in between the powers of 2.

④

Suppose that $\forall\, a_1, a_2, \ldots, a_{K+1}$

$$\left(a_1 a_2 \ldots a_{K+1}\right)^{\frac{1}{K+1}} \overset{(*)}{\leq} \frac{a_1 + a_2 + \ldots + a_{K+1}}{K+1}$$

Can we deduce that $(*)$ holds w/ $K+1$ replaced by $K$?

In $(*)$, choose $a_{K+1} = S_K = \dfrac{a_1 + a_2 + \ldots + a_K}{K}$

$$\left(a_1 a_2 \ldots a_K S_K\right)^{\frac{1}{K+1}} \leq S_K$$

$$\left(a_1 a_2 \ldots a_K\right)^{\frac{1}{K+1}} \leq S_K^{1-\frac{1}{K+1}} = S_K^{\frac{K}{K+1}}$$

$$\left(a_1 a_2 \ldots a_K\right)^{\frac{1}{K}} \leq S_K = \frac{a_1 + a_2 + \ldots + a_K}{K}$$

(5)

$$F(R) = \int_a^b e^{iRf(x)}\,dx$$

Does $F(R) \to 0$ as $R \to \infty$?

$$\int_a^b \left(e^{iRf(x)}\right)' \frac{dx}{iRf'(x)} \sim \text{hypothesis } f'(x) \geq 1$$

$$\parallel$$

$$\underbrace{\left.\frac{e^{iRf(x)}}{iRf'(x)}\right|_a^b}_{\text{I}} - \underbrace{\int_a^b e^{iRf(x)} \frac{d}{dx}\left(\frac{1}{iRf'(x)}\right) dx}_{\text{II}}$$

$$|\text{I}| \leq \frac{1}{R} + \frac{1}{R} = \frac{2}{R}$$

$$\underline{II} = \int_a^b e^{iRg(x)} \frac{d}{dx}\left(\frac{1}{iRg'(x)}\right) dx$$

$$|\underline{II}| \le \int_a^b \left|\frac{d}{dx}\underbrace{\left(\frac{1}{iRg'(x)}\right)}_{\text{assume that } g' \nearrow}\right| dx$$

$$|\underline{II}| \le \left| \int_a^b \frac{d}{dx}\left(\frac{1}{iRg'(x)}\right) dx \right|$$

$$= \left| \frac{1}{iRg'(x)} \Bigg|_a^b \right| \le \frac{1}{R}\left(\frac{1}{g'(b)} - \frac{1}{g'(a)}\right)$$

$$\le \frac{2}{R}$$

7

Van der Corput lemma: $f \in C^1(a,b)$ $f'$ monotonic

$f'(x) \geq 1$ on $[a,b]$

Then $\left| \int_a^b e^{iRf(x)}\, dx \right| \leq \frac{4}{R}$