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Basic inequalities:

$a_i \in \mathbb{R} \quad b_i \in \mathbb{R}$

Cauchy-Schwarz:

$$\left|\sum_i a_i b_i\right| \leq \left(\sum_i a_i^2\right)^{\frac{1}{2}} \left(\sum_i b_i^2\right)^{\frac{1}{2}} \quad (*)$$

Let $A = \left(\sum_i a_i^2\right)^{\frac{1}{2}} \quad B = \left(\sum_i b_i^2\right)^{\frac{1}{2}}$

$(*)$ becomes

$$\sum_i \frac{a_i}{A} \frac{b_i}{B} \leq 1$$

Observe that

$$\sum_i \left(\frac{a_i}{A}\right)^2 = \sum_i a_i^2 / A^2 = 1$$

Similarly, $\sum_i \left(\frac{b_i}{B}\right)^2 = 1.$

We have thus reduced matters to showing that

$$\sum_i a_i b_i \leq 1 \text{ if } \sum a_i^2 = \sum_i b_i^2 = 1.$$

(2)

Observe that

$$a_i b_i \leq \frac{a_i^2 + b_i^2}{2} \quad \text{since}$$

$$0 \leq (a_i - b_i)^2 = a_i^2 + b_i^2 - 2a_i b_i$$

It follows that

$$\sum a_i b_i \leq \frac{1}{2}\sum_i a_i^2 + \frac{1}{2}\sum_i b_i^2$$

$$= \frac{1}{2} + \frac{1}{2} = 1.$$

The following example illustrates the power of Cauchy - Schwarz:

$$\begin{pmatrix} 1 & 0 & \cdots & & 0 \\ & 1 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & & 0 \\ & 1 & & & 1 \end{pmatrix}$$

not allowed!

no rectangles w/ 1's at the vertices.

(3)

More precisely, let $\{a_{ij}\}_{i,j=1}^{n}$ be an $n \times n$ matrix, where $a_j = 0$ or 1.

The "rectangle" condition means that

if $j \neq j'$, $\boxed{a_{ij} \cdot a_{ij'} = 1}$ for at most one value of $\underline{\underline{i}}$.

Question: How many 1's can our matrix possibly have?

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\right)^2 = \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\right) \cdot 1\right)^2$$

$$\leq \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\right)^2 \cdot \sum_{i=1}^{n} 1^2$$

$$= n \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{j'=1}^{n} a_{ij}\, a_{ij'}$$

In order to use our condition

$a_{ij} \cdot a_{ij'} = 1$ for at most one value of i, we must have $j \neq j'$

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$$n\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{j'=1}^{n} a_{ij}a_{ij'} = n\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{2} + n\sum_{i=1}^{n}\sum_{j\neq j'} a_{ij}a_{ij'}$$

$j=j'$ $\qquad\qquad$ $j\neq j'$

$$= \underline{I} + \underline{II}$$

$\underline{I} \leq n^3$

To estimate $\underline{II}$, note that since $j \neq j'$, there is at most <u>one</u> $i \ni a_{ij}\cdot a_{ij'} = \underline{1}$

It follows that $\underline{II} \leq n\cdot n(n-1) \leq n^3$.

We conclude that

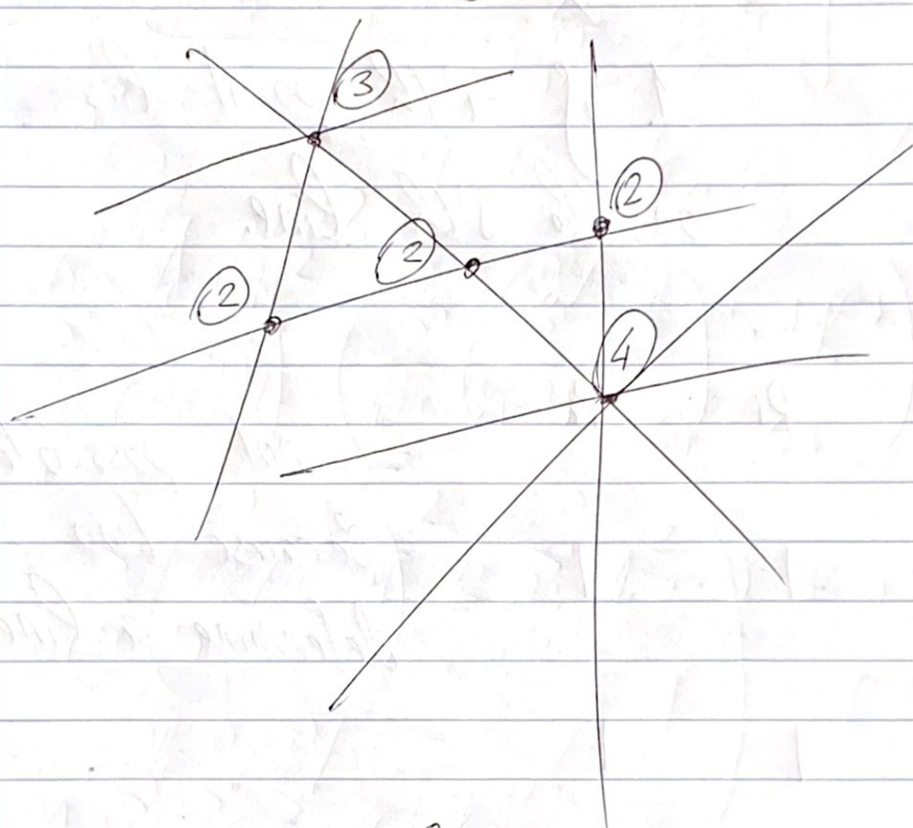
$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\right)^2 \leq 2n^3, \text{ so}$$

$$\sum_{i,j=1}^{n} a_{ij} \leq \sqrt{2}\, n^{\frac{3}{2}}$$

Exercise: For a sequence of n's $\longrightarrow \infty$, construct an $n \times n$ matrix satisfying the rectangle condition and containing

$$\approx n^{\frac{3}{2}} \ \underline{1\text{'s}}.$$

Where does the rectangle condition come from?



n points, n lines

incidence = (point, line) $\ni$ point $\in$ line

I = total number of incidences

(6)

$$
\begin{array}{c|ccccc}
 & p_1 & p_2 & \cdots & p_i & \cdots & p_n \\
\hline
l_1 & & & & & & \\
l_2 & & & & & & \\
\vdots & & & & & & \\
l_j & & & & \boxed{1} & & \\
 & & & & & & \\
l_n & & & & & &
\end{array}
$$

$p_1\ p_2 \ldots p_i \ldots p_n$ ~ points

lines

$\underline{1}$ if i'th point lies on the j'th line.

$$
\begin{array}{c|cc}
 & p_i & p_{i'} \\
\hline
l_j & 1 & 1 \\
l_{j'} & 1 & 1
\end{array}
$$

not possible because two points determine a line!

Holder's inequality: $1 < p, q < \infty$ $\frac{1}{p} + \frac{1}{q} = 1$
$a_i, b_i \in \mathbb{R}$

Then $\left| \sum_i a_i b_i \right| \leq \underbrace{\left(\sum_i |a_i|^p \right)^{\frac{1}{p}}}_{A} \underbrace{\left(\sum_i |b_i|^q \right)^{\frac{1}{q}}}_{B}$

Proof:

Applying the same idea as in the proof of Cauchy-Schwarz, we divide both sides by AB and reduce matters to proving that

$\left| \sum a_i b_i \right| \leq 1$ if $A = B = 1$

What is the analog of the inequality

$ab \leq \frac{a^2 + b^2}{2}$ we used for Cauchy-Schwarz?

If we were to mimick the proof of C-S, we would need the inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b \geq 0 \quad \frac{1}{p} + \frac{1}{q} = 1$$

But is it true?

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The fact that $\frac{1}{p}+\frac{1}{q}=1$ gives us a clue that convexity is involved, so we rewrite

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q} \quad \text{in the form}$$

$$ab \leq (1-t)a^p + t b^q$$

Let $a = e^{(1-t)x}$ $b = e^{ty}$ for some x, y

Then the inequality takes the form

$$e^{(1-t)x+ty} \leq (1-t)e^x + te^y,$$

which is just a statement that the exponential function is convex!

How do we really know that the exponential function is convex? We shall address this point comprehensively in a moment.

(9)

Let us first complete the proof of Hölder.

We must show that

$$\left|\sum_i a_i b_i\right| \leq 1 \text{ if } \sum_i |a_i|^p = \sum_i |b_i|^q = 1.$$

$$\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p, q < \infty$$

We have

$$\sum_i a_i b_i \leq \frac{1}{p}\sum_i |a_i|^p + \frac{1}{q}\sum_i |b_i|^q$$

$$= \frac{1}{p} + \frac{1}{q} = 1 \checkmark$$

Let us now explore convexity a bit more.

Let φ be a twice differentiable function such that $\varphi''(x) \geq 0$.

Is it true that $(x < y)$

$$\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y)?$$

(10)

Let

$$F(t) = (1-t)\varphi(x) + t\varphi(y) - \varphi((1-t)x+ty)$$

$$F'(t) = \varphi(y) - \varphi(x) - (y-x)\varphi'((1-t)x+ty)$$

$$F''(t) = -(y-x)^2\varphi''((1-t)x+ty) \leq 0$$

$F(0) = F(1) = 0 \qquad F''(t) \leq 0$

Does this imply that $F \geq 0$?

which is what we want

Suppose not! Then $\exists\ t_0 \ni F(t_0) < 0,$

$t_0 \in (0, 1)$

By Mean Value Theorem, $\exists\ c_0 \in (0, t_0) \ni$

$$\underbrace{F(t_0) - F(0)}_{\text{negative}} = (t_0 - 0)F'(c_0) \longrightarrow F'(c_0) < 0$$

Similarly, $\exists\ c_0' \ni$

$$\underbrace{F(1)}_{0} - \underbrace{F(t_0)}_{\text{positive}} = (1-t_0)F'(c_0')$$

$$\Longrightarrow F'(c_0') > 0$$

Applying MVT yet again, $\exists\ c_1 \in (c_0, c_0')$

$\ni\ F'(c_0') - F'(c_0) = (c_0' - c_0)F''(c_1)$

$$\Longrightarrow F''(c_1) > 0$$

CONTRADICTION!