Hitting Time Distributions of Random Walks on Finite Graphs

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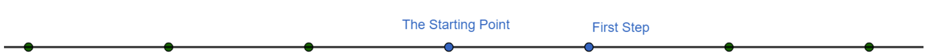
Introduction

Random walks are a fundamental topic in probability theory with wide-ranging applications across physics, engineering, mathematics, and computer science. In fact, there are over 11,000 papers on arXiv that explore them in depth. Random walks can be used to model a variety of real-world phenomena, including fluid dynamics, stock price movements, genetic drift, animal foraging behavior, and search algorithms.

But what exactly is a random walk? At its core, it's a simple model of an indecisive traveler. To understand how such a versatile model works, let's look at its simplest form. Imagine standing at 0 on a number line. You flip a fair coin: if it lands heads, you take a step to the right; if tails, a step to the left. For instance, let's say you flip the coin and get heads—you move to position 1. That's the first step in your random walk.



and if we get tails after that...

A lot of research goes into modeling where we would end up after n steps and with what probability. This process can also be generalized to higher dimensions (moving on a grid or in a 3d space instead of just the number line where we move in each direction with the same probability) and those processes have their own interesting properties. For the purpose of this paper, our random walks will occur on graphs.

A graph is a mathematical structure made up of nodes, called vertices, and the connections between them, called edges. This can be conceived as a system of towns and the roads that connect them. Consider now a random walk on a graph. At each step, the choice of direction depends only on the current position and the available edges, not on the path taken to get there. Whether the edges are directed or undirected, weighted or unweighted, the behavior of the random walk changes accordingly, revealing insights into the structure and dynamics of the graph itself. Through this lens, a graph becomes not just a static diagram, but a playground of movement, chance, and connectivity.

Motivation and Problem Statement

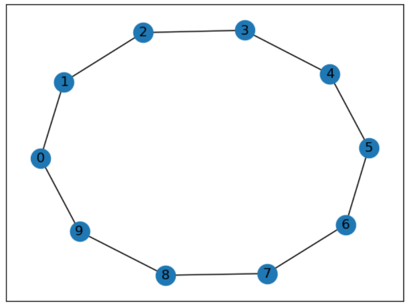
The central focus of this honors thesis is the *hitting time*, which refers to the number of steps required for a random walk to move from one vertex to another. In probabilistic terms, for a graph $G = (V, E)$, we define the hitting time as

$$\tau_{i,j} = \inf\{t \mid X_0 = i, X_t = j\}$$

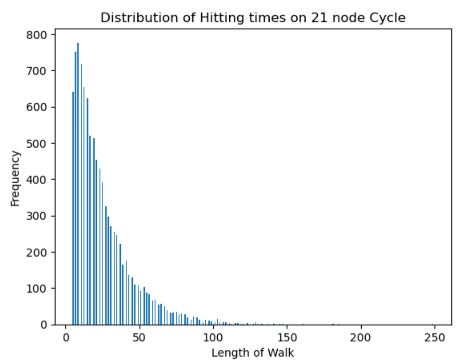
where X_t denotes the position of the random walker at time t, and i and j are two distinct vertices in the graph.

Studying hitting times in Markov chains is essential because they provide deep insights into the behavior and dynamics of stochastic processes over time. Specifically, a hitting time represents the expected number of steps it takes for a Markov chain to reach a particular state for the first time, starting from a given initial state. This concept is crucial across a variety of fields. In network theory, it helps us understand the expected time it takes for a random walk to reach a specific node. In reliability engineering, it can model the time until a system fails or recovers. In algorithm design, hitting times are important for evaluating the efficiency of randomized algorithms, while in economics and biology, they offer tools for modeling transitions between key states or events. By studying hitting times, we gain a clearer understanding of the temporal aspects of randomness, enabling us to predict, optimize, and control complex systems governed by uncertainty. Any system that can be modeled by a random walk benefits from the investigation of hitting times.

One effective way to explore hitting times is through simulation. For example, consider the cycle graph with 10 nodes. The following diagram shows the graph:

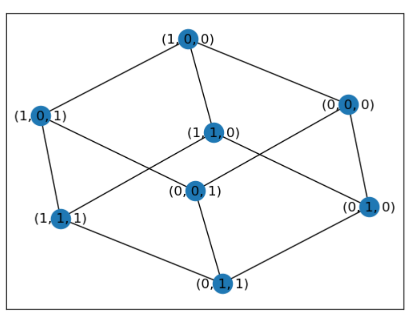


Suppose we are interested in the random variable $\tau_{0,5}$, representing the number of steps needed to go from node 0 to node 5. After running 10,000 trials, we obtain the following frequency graph:

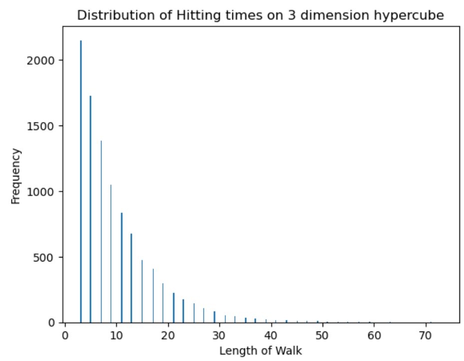


What stands out here is the *large variance* in the distribution. The sample mean is 25.0306, and the sample variance is 410, indicating that the variance is relatively large compared to the mean.

We can observe a similar behavior in the hypercube graph with 8 nodes, which is shown below:



Now, suppose we are interested in $\tau_{(0,0,0),(1,1,1)}$, the number of steps needed to move from node $(0, 0, 0)$ to node $(1, 1, 1)$. After running 10,000 trials, we obtain the following frequency graph:



The sample mean in this case is 10, and the sample variance is 63.

This observation leads to the central motivation for this thesis. In the literature, the primary focus in the study of hitting times has been on *expected hitting times*, denoted $E[\tau_{i,j}]$. However, as demonstrated above, the expected value alone is often a poor predictor of how the random walk behaves. In this paper, we will examine two related quantities: $P(\tau_{i,j} = n)$, the probability mass function of hitting times, and $\text{Var}(\tau_{i,j})$, the variance of the hitting time. Despite their importance, these quantities receive less attention due to computational challenges. Our approach will involve general computations, with a subsequent focus on vertex-transitive graphs.

Markov Chains on General Graphs

Many of the ideas from the next section comes from [4] and [5]

Distributions

Let's try to find the distribution of $P(\tau_{i,j} = n)$ on our graph G with Markov matrix A. One relationship becomes clear.

$$P(\tau_{i,j} = n) = \sum_{k \neq j} P(\tau_{i,k} = 1) P(\tau_{k,j} = n - 1)$$

The above formula calculates the probability of reaching an adjacent node to our ending node in $n-1$ steps and then making a step from that adjacent node to the end. By setting $k \neq j$, we make sure that we aren't adding the probability that we arrive to ending node j one move early. The above is true, as each step of a random walk is independent. Let us fix an ending node j, and then let us define a vector

$$P_n = \begin{bmatrix} P(\tau_{1,j} = n) \\ P(\tau_{2,j} = n) \\ \vdots \\ P(\tau_{|V|-1,j} = n) \end{bmatrix}$$

the nodes $1, 2...|V|-1$ represent some arbitrary numbering of the nodes of the graph once j is removed. Let Q be the matrix such that $Q_{ik} = P(\tau_{ik} = 1)$ such that $i, k \neq j$. We then have

$$P_n = QP_{n-1}$$

as the recursion above is simply matrix multiplication. So then by induction, we have

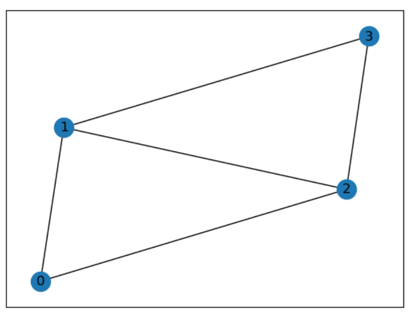
$$P_n = Q^{n-1}P_1$$

One might notice that Q is simply the Markov matrix of our graph but with removed jth row and jth column. Therefore, here we have a solid way of calculating distributions. As we are taking arbitrary powers of a matrix, it often comes down to diagonalizing Q. This is often hard to do by hand, but a computer can help. There are a couple things to note about the eigenvalues of such a matrix.

1. Q is a substochastic matrix as the sum of every row is less than or equal to 1. This implies that $|\lambda| < 1$. Where λ is an eigenvalue of 1.

2. Q is the adjacency graph of a subgraph of G. Which implies by the Interlacing theorem of Spectral Graph Theory that all the eigenvalues of Q are embedded between the eigenvalues of A.

Consider this graph



. We will use every step on a random walk is independent and assuming that walking across any edge is equally likely. Setting $j = 0$ and then Q is

$$\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

and P_1 is

$$\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

. Then

$$Q^{n-1}P_1 = \begin{bmatrix} P(\tau_{1,0} = n) \\ P(\tau_{2,0} = n) \\ P(\tau_{3,0} = n) \end{bmatrix} = \begin{bmatrix} \frac{-(-1)^n*2^{2n}*3^n*\sqrt{13}+\left(2\sqrt{13}+14\right)^n*\sqrt{13}}{13*\left(\sqrt{13}+1\right)^n 2^n 3^n} \\ \frac{-(-1)^n*2^{(2*n)}*3^n*\sqrt{13}+\left(2\sqrt{13}+14\right)^n\sqrt{13}}{13*\left(\sqrt{13}+1\right)^n 2^n 3^n} \\ \frac{13*(-1)^n*2^{2n}3^n+13\left(2\sqrt{13}+14\right)^n+(-1)^n 2^{2n}3^n\sqrt{13}-\left(2\sqrt{13}+14\right)^n\sqrt{13}}{26*\left(\sqrt{13}+1\right)^n 2^n 3^n} \end{bmatrix}$$

We can see that even for relatively simple looking graphs, the distributions can be very complicated and often intractable to compute by hand with larger graphs. Later, we will restrict the graphs we will work with to make sure this process is simpler.

Characteristic Function

Let's consider the value of the $\phi_{\tau_{i,j}}(t)$. We can compute this directly using standard techniques

$$\phi_{\tau_{i,j}}(t) = E[e^{it\tau_{i,j}}] = \sum_{n=-\infty}^{\infty} e^{int}P(\tau_{i,j} = n)$$

As it is impossible to move to another node in negative moves. Also since $i \neq j$, it follows that $n \neq 0$. Therefore, we can write the sum as

$$\sum_{n=1}^{\infty} e^{int}P(\tau_{i,j} = n)$$

$$\sum_{n=1}^{\infty} e^{int}e_i^T Q^{n-1}P_1$$

Instead of just considering just one moment generating function, we consider all the characteristic functions $\phi_{\tau_{i,j}}(t)$ for all $i \neq j$. We will call this vector function $M_{\tau_{*,j}}(t)$.

$$\phi_{\tau_{*,j}}(t) = \sum_{n=1}^{\infty} e^{int}Q^{n-1}P_1$$

$$\phi_{\tau_{*,j}}(t) = e^{it}\sum_{n=0}^{\infty} e^{int}Q^{n}P_1$$

$$\phi_{\tau_{*,j}}(t) = e^{it}(I - e^{it}Q)^{-1}P_1$$

We can take the derivative of this

$$\phi'_{\tau_{*,j}}(t) = ie^{it}(I - e^{it}Q)^{-1}P_1 + e^{it}(I - e^{it}Q)^{-1}ie^{it}Q(I - e^{it}Q)^{-1}P_1$$

$$\phi'_{\tau_{*,j}}(t) = ie^{it}(I - e^{it}Q)^{-1}(I + e^{it}Q(I - e^{it}Q)^{-1})P_1$$

$$\phi'_{\tau_{*,j}}(t) = ie^{it}(I - e^{it}Q)^{-1}((I - e^{it}Q + e^{it}Q)(I - e^{it}Q)^{-1})P_1$$

$$\phi'_{\tau_{*,j}}(t) = ie^{it}(I - e^{it}Q)^{-2}P_1$$

Since $iE[\tau_{*,j}] = \phi_{\tau_{*,j}}(0)$

$$\phi'_{\tau_{*,j}}(0) = i(I - Q)^{-2}P_1$$

Therefore, we have

$$E[\tau_{*,j}] = (I - Q)^{-2}P_1$$

We have another way of reaching this quantity by considering the quantity agian.

$$\phi_{\tau_{*,j}}(t) = e^{it}P_1$$

$$(I - e^{it}Q)\phi_{\tau_{*,j}}(t) = e^{it}P_1$$

We take the derivative of both sides

$$(I - e^{it}Q)\phi'_{\tau_{*,j}}(t) - ie^{it}Q\phi_{\tau_{*,j}}(t) = ie^{it}P_1$$

$$(I - e^{it}Q)\phi'_{\tau_{*,j}}(t) = ie^{it}P_1 + ie^{it}Q\phi_{\tau_{*,j}}(t)$$

We now plug in $t = 0$.

$$(I - Q)\phi'_{\tau_{*,j}}(0) = iP_1 + iQ\phi_{\tau_{*,j}}(0)$$

Since $\phi_{\tau_{i,j}}(0) = \sum_{n=1}^{\infty} e^{i0t}P(\tau_{i,j} = n) = 1$. If $\mathbf{1}$ is the vector of all 1's, then we have that

$$(I - Q)\phi'_{\tau_{*,j}}(0) = iP_1 + Q\mathbf{1}$$

$$\phi'_{\tau_{*,j}}(0) = i(I - Q)^{-1}(P_1 + Q\mathbf{1})$$

It follows that $Q\mathbf{1}_i = \sum_{k \neq j} P(\tau_{i,k} = 1)$. Therefore, $(P_1 + Q\mathbf{1}) = \mathbf{1}$

$$\phi'_{\tau_{*,j}}(0) = i(I - Q)^{-1}\mathbf{1}$$

So then

$$E[\tau_{*,j}] = (I - Q)^{-1}\mathbf{1} = \sum_{n=0}^{\infty} Q^n\mathbf{1}$$

We then have that second derivative of this is

$$\phi''_{\tau_{*,j}}(t) = -2Q(I - Q)^{-2}\mathbf{1}$$

Which means that

$$E[\tau_{*,j}^2] = 2Q(I - Q)^{-2}\mathbf{1}$$

$$E[\tau_{*,j}^2] = 2\sum_{n=0}^{\infty} nQ^n\mathbf{1}$$

When it comes to actually computing these quantities. The above calculations exist for approximate values found by a computer. In some cases, it is feasible to use the above formulae to find the distributions for general classes of graphs. The cases below are generally easier to compute. Later in this paper, we will introduce machinery to tackle harder cases.

Complete Case

The complete graph is where each node is connected to every other node. It looks like this. In this case, it is clear that for a complete graph of k nodes, we have

$$P(\tau_{i,j} = 1) = \frac{1}{k-1}$$

if we have a simple random walk. Then, it is clear that for all vertices i, j in the graph

$$P(\tau_{i,j} = n) = \left(\frac{k-2}{k-1}\right)^{n-1} \frac{1}{k-1}$$

Where the walker makes $n-1$ "wrong" moves and then makes the right one once. We can see above that the distribution is geometric with success probability of $\frac{1}{k-1}$ so

$$E[\tau_{i,j}] = k - 1$$

$$Var[\tau_{i,j}] = (k-1)(k-2)$$

Complete Bipartite Case

A complete bipartite graph consists of two disjoint sets of nodes, A and B, of sizes k_1 and k_2, respectively. Every node in A is connected to every node in B, and there are no edges within A or within B.

Suppose a random walker starts at a node in A and aims to reach a node in B. Due to the bipartite structure, the walker alternates between A and B on each step. That is, all paths from A to B (and vice versa) have odd lengths, while paths from A to A (or B to B) have even lengths.

Let $i \in A$ and $j \in B$. The probability that the walker moves directly to node j in one step is

$$\mathbb{P}(\tau_{i,j} = 1) = \frac{1}{k_2},$$

since the walker chooses uniformly among the k_2 neighbors in B.

If the walker does not reach j on the first attempt, it returns to some node in A, then tries again, and so on. Each failed attempt consists of two steps: $A \to B \to A$, and succeeds with probability $1/k_2$ on the next $A \to B$ step. Thus, the number of such trials until success follows a geometric distribution with success probability $1/k_2$, but each trial takes 2 steps after the first.

Therefore, for $n \geq 1$:

$$\mathbb{P}(\tau_{i,j} = 2n - 1) = \left(1 - \frac{1}{k_2}\right)^{n-1} \cdot \frac{1}{k_2}, \quad \text{for } i \in A,\ j \in B.$$

This is a geometric distribution over odd steps.

Hence, the expected hitting time is:

$$\mathbb{E}[\tau_{i,j}] = \sum_{n=1}^{\infty} (2n-1) \cdot \left(1 - \frac{1}{k_2}\right)^{n-1} \cdot \frac{1}{k_2} = 2k_2 - 1.$$

For $i, j \in A$ with $i \neq j$, the walker must first move to B, and then to j in A, which takes at least two steps. This again produces a geometric-like distribution (over even steps), with the same success probability:

$$\mathbb{P}(\tau_{i,j} = 2n) = \left(1 - \frac{1}{k_2}\right)^{n-1} \cdot \frac{1}{k_2}, \quad \text{for } i, j \in A, \ i \neq j.$$

Hence,

$$\mathbb{E}[\tau_{i,j}] = \sum_{n=1}^{\infty} 2n \cdot \left(1 - \frac{1}{k_2}\right)^{n-1} \cdot \frac{1}{k_2} = 2k_2.$$

For the variance, note that if X is a geometric random variable with success probability $p = \frac{1}{k_2}$, then:

$$\mathbb{E}[X] = \frac{1}{p} = k_2$$

$$\mathrm{Var}(X) = \frac{1-p}{p^2} = k_2(k_2 - 1)$$

The hitting time from A to B is $\tau_{i,j} = 2X - 1$, so:

$$\mathbb{E}[\tau_{i,j}] = 2\mathbb{E}[X] - 1 = 2k_2 - 1, \quad \mathrm{Var}(\tau_{i,j}) = 4\mathrm{Var}(X) = 4k_2(k_2 - 1).$$

Similarly, for $i, j \in A$, $\tau_{i,j} = 2X$, so:

$$\mathbb{E}[\tau_{i,j}] = 2k_2, \quad \mathrm{Var}(\tau_{i,j}) = 4k_2(k_2 - 1).$$

Thus, both cases have the same variance. The only difference lies in the offset of 1 step in expectation between cross-set and within-set hitting times.

Cycle Case

Let's perform these computations for a cycle. It clearly follows for a k-cycle (denoted as C_k)

$$Q = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & ... & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & ... & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & \frac{1}{2} & 0 \end{bmatrix}, P_1 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

So then we have

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & ... & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & ... & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} P(\tau_{1,0} = n-1) \\ P(\tau_{2,0} = n-1) \\ ... \\ P(\tau_{k-1,0} = n-1) \end{bmatrix} = \begin{bmatrix} P(\tau_{1,0} = n) \\ P(\tau_{2,0} = n) \\ ... \\ P(\tau_{k-1,0} = n) \end{bmatrix}$$

Or in other words

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & ... & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & ... & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & \frac{1}{2} & 0 \end{bmatrix}^{n-1} \begin{bmatrix} P(\tau_{1,0}=1) \\ P(\tau_{2,0}=1) \\ ... \\ P(\tau_{k-1,0}=1) \end{bmatrix} = \begin{bmatrix} m_n(1) \\ m_n(2) \\ ... \\ m_n(k-1) \end{bmatrix}$$

So it follows we want to take the diagonalization of the Toeplitz Matrix above. Let us call that matrix H and its diagonlization LDL^{-1}. Thankfully, the eigenvectors and eigenvalues for tridiagonal toeplitz matrices are well known and with that we have the following as the diagonalization. These eigenvalues were found in [1]

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & ... & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & ... & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & \frac{1}{2} & 0 \end{bmatrix}$$

$$= \frac{2}{k} \begin{bmatrix} \sin(\frac{\pi}{k}) & \sin(\frac{2\pi}{k}) & ... & \sin(\frac{(k-1)\pi}{k}) \\ \sin(\frac{2\pi}{k}) & \sin(\frac{4\pi}{k}) & ... & \sin(\frac{2(k-1)\pi}{k}) \\ ... & ... & ... & ... \\ \sin(\frac{(k-1)\pi}{k}) & \sin(\frac{2(k-1)\pi}{k}) & ... & \sin(\frac{(k-1)^2\pi}{k}) \end{bmatrix} \begin{bmatrix} \cos(\frac{\pi}{k}) & & 0 & ... & 0 \\ 0 & \cos(\frac{2\pi}{k}) & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \cos(\frac{(k-1)\pi}{k}) \end{bmatrix}$$

$$\begin{bmatrix} \sin(\frac{\pi}{k}) & \sin(\frac{2\pi}{k}) & ... & \sin(\frac{(k-1)\pi}{k}) \\ \sin(\frac{2\pi}{k}) & \sin(\frac{4\pi}{k}) & ... & \sin(\frac{2(k-1)\pi}{k}) \\ ... & ... & ... & ... \\ \sin(\frac{(k-1)\pi}{k}) & \sin(\frac{2(k-1)\pi}{k}) & ... & \sin(\frac{(k-1)^2\pi}{k}) \end{bmatrix}$$

As we know from earlier,

$$\begin{bmatrix} P(\tau_{1,0}=1) \\ P(\tau_{2,0}=1) \\ ... \\ P(\tau_{k-1,0}=1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ ... \\ \frac{1}{2} \end{bmatrix}$$

So, then we first multiply by L^{-1} or the last matrix in the diagonalization.

$$\begin{bmatrix} \sin(\frac{\pi}{k}) & \sin(\frac{2\pi}{k}) & ... & \sin(\frac{(k-1)\pi}{k}) \\ \sin(\frac{2\pi}{k}) & \sin(\frac{4\pi}{k}) & ... & \sin(\frac{2(k-1)\pi}{k}) \\ ... & ... & ... & ... \\ \sin(\frac{(k-1)\pi}{k}) & \sin(\frac{2(k-1)\pi}{k}) & ... & \sin(\frac{(k-1)^2\pi}{k}) \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ ... \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sin(\frac{\pi}{k}) + \sin(\frac{(k-1)\pi}{k}) \\ \sin(\frac{2\pi}{k}) + \sin(\frac{2(k-1)\pi}{k}) \\ ... \\ \sin(\frac{(k-1)\pi}{k}) + \sin(\frac{(k-1)^2\pi}{k}) \end{bmatrix}$$

We next multiply by the diagonal matrix.

$$\frac{1}{2} \begin{bmatrix} \cos(\frac{\pi}{k}) & & 0 & ... & 0 \\ 0 & \cos(\frac{2\pi}{k}) & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \cos(\frac{(k-1)\pi}{k}) \end{bmatrix}^{n-1} \begin{bmatrix} 0 \\ \sin(\frac{2\pi}{k}) \\ 0 \\ \sin(\frac{4\pi}{k}) \\ ... \end{bmatrix} = \frac{1}{2} \begin{bmatrix} cos(\frac{\pi}{k}) \sin(\frac{\pi}{k})^{n-1} + \sin(\frac{(k-1)\pi}{k})) \\ cos(\frac{2\pi}{k})^{n-1} \sin(\frac{2\pi}{k}) + \sin(\frac{2(k-1)\pi}{k}) \\ ... \\ cos(\frac{(k-1)\pi}{k})^{n-1} (\sin(\frac{(k-1)\pi}{k}) + \sin(\frac{(k-1)^2\pi}{k})) \end{bmatrix}$$

And lastly multiplying by L.

$$\frac{2}{k}\begin{bmatrix} \sin(\frac{\pi}{k}) & \sin(\frac{2\pi}{k}) & ... & \sin(\frac{(k-1)\pi}{k}) \\ \sin(\frac{2\pi}{k}) & \sin(\frac{4\pi}{k}) & ... & \sin(\frac{2(k-1)\pi}{k}) \\ ... & ... & ... & ... \\ \sin(\frac{(k-1)\pi}{k}) & \sin(\frac{2(k-1)\pi}{k}) & ... & \sin(\frac{(k-1)^2\pi}{k}) \end{bmatrix} \frac{1}{2}\begin{bmatrix} cos(\frac{\pi}{k})\sin(\frac{\pi}{k})^{n-1} + \sin(\frac{(k-1)\pi}{k})) \\ cos(\frac{2\pi}{k})^{n-1}\sin(\frac{2\pi}{k}) + \sin(\frac{2(k-1)\pi}{k}) \\ ... \\ cos(\frac{(k-1)\pi}{k})^{n-1}(\sin(\frac{(k-1)\pi}{k}) + \sin(\frac{(k-1)^2\pi}{k})) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{k}\sum_{j=0}^{k-1}\cos(\frac{j\pi}{k})^{n-1}(\sin(\frac{j\pi}{k}) + \sin(\frac{j(k-1)\pi}{k}))\sin(\frac{j\pi}{k}) \\ \frac{1}{k}\sum_{j=0}^{k-1}\cos(\frac{j\pi}{k})^{n-1}(\sin(\frac{j\pi}{k}) + \sin(\frac{j(k-1)\pi}{k}))\sin(\frac{2j\pi}{k}) \\ \frac{1}{k}\sum_{j=0}^{k-1}\cos(\frac{j\pi}{k})^{n-1}(\sin(\frac{j\pi}{k}) + \sin(\frac{j(k-1)\pi}{k}))\sin(\frac{3j\pi}{k}) \\ ... \\ \frac{1}{k}\sum_{j=0}^{k-1}\cos(\frac{j\pi}{k})^{n-1}(\sin(\frac{j\pi}{k}) + \sin(\frac{j(k-1)\pi}{k}))\sin(\frac{(k-1)j\pi}{k}) \end{bmatrix}$$

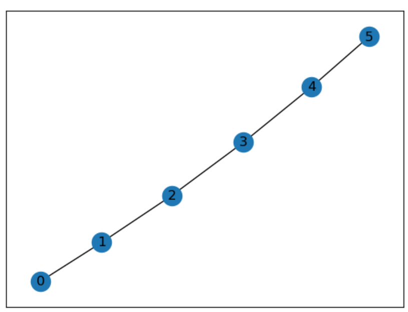
Therefore

$$P(\tau_{i,j} = n) = \frac{1}{k}\sum_{j=0}^{k-1}\cos(\frac{j\pi}{k})^{n-1}(\sin(\frac{j\pi}{k}) + \sin(\frac{j(k-1)\pi}{k}))\sin(\frac{ij\pi}{k})$$

We will discuss the expectation and variance of this distribution later.

Path Graphs

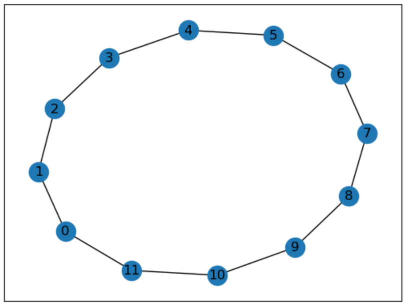
As we explored above, one important class of graphs where hitting time distributions can be derived explicitly is the cycle. Consider the path graph P_k, which is a chain of k nodes connected in sequence.



For generality, we take one of the endpoints as the absorbing state (i.e., the destination node).

This assumption does not result in a loss of generality. Suppose instead we selected an interior node (e.g., node 3 in a 5-node path) as the destination. Then, any walk from one end (e.g., node 1) that eventually reaches the other end (e.g., node 5) without hitting the interior node is irrelevant to the hitting time to that interior node. Thus, conditioning on reaching the interior node without escape effectively partitions the walk into two disjoint path subgraphs. Hence, the analysis of hitting times to an endpoint captures the general behavior.

Now consider the cycle graph C_{2k}, where nodes are connected in a closed loop of $2k$ vertices. By folding the cycle along the line of symmetry, we can pair nodes that are equidistant from a chosen target node. (The line of symmetry would be a hypothetical line from 0 to 6.



Under this pairing, the probability of stepping toward or away from the target remains unchanged, and the resulting random walk is statistically identical (in terms of hitting time) to that of a path graph with k nodes. Therefore, the hitting time distribution for a path of k nodes corresponds exactly to that of a cycle of $2k$ nodes under this symmetry.

Cayley Graphs

Let's suppose we have a random walk on a group G. We can imagine each element in our group $g \in G$ being associated with a node in a graph. Let's suppose that two nodes g, h are connected if for some $c \in C \subset G$, $gc = h$. We often call C the connection set. The resultant graph is called a Cayley graph. Many times we want to consider a graph connected only by its generators. This changes for different kinds of graphs. If we want to perform a random walk on such a graph, we assign a probability from moving from $g \to gc$. Let us call this $p(c)$. Generally, C is symmetric. This means that if $x \in C$ then $x^{-1} \in C$. If $p(x) = p(x^{-1})$, then we have a symmetric random walk.

In the context of Cayley graphs, the cycle graph on k nodes can be seen as the Cayley graph of the cyclic group $\mathbb{Z}_k$ with generating set $\{\pm 1\}$. Extending this idea, we now consider the group $\mathbb{Z}_p^2$, where p is an odd prime. A standard set of generators for its Cayley graph is:

$$\{(\pm 1, 0), (0, \pm 1)\},$$

which gives rise to the 2D torus graph — a grid with wrap-around edges.

However, an alternative set of generators is:

$$\{(\pm 1, \pm 1)\},$$

which corresponds to diagonal steps in the lattice. These generators also produce a valid Cayley graph, albeit with a rotated geometry. However, why would we do this? This shift let's us consider random walks that are independent in both coordinates, something that is not true if we are moving along the standard generators of Z_p^2. This happens as the probability of moving vertically is inevitably tied to the probability of moving to the side as a walker must pick one or the other. By considering a diagonal random walk, with the above generators, we introduce independence in the walks in either of the coordinates. Importantly, these generator sets are related via a linear automorphism:

$$\phi(a, b) = \left(\frac{a+b}{2}, \frac{a-b}{2} \right),$$

which is an automorphism of $\mathbb{Z}_p^2$ since 2 is invertible in $\mathbb{Z}_p$ (as p is odd). This transformation maps the standard coordinate basis to the new one spanned by the diagonal generators.

Let $c_n(i)$ denote the probability that a walk on a p-cycle starting at 0 hits i for the first time at step n. If we define $\phi^{-1}(a-c, b-d) = (a', b')$, then (a', b') represents the displacement between (a, b) and (c, d) expressed in terms of the diagonal generator basis. Under the assumption of independence in each coordinate (which holds due to the structure of the walk), the hitting time distribution on $\mathbb{Z}_p^2$ can be expressed as:

$$P(\tau_{(a,b),(c,d)} = n) = \sum_{i=0}^{n} c_i(a') \cdot c_{n-i}(b').$$

That is, the distribution of the hitting time is a convolution of the 1D hitting time distributions along the transformed coordinates. The reason that Z_p^2 was singled out instead of the direct products of other cyclic groups is that the ϕ is an automorphism on Z_p^2 and not in other such groups. However, if one was to draw those graphs generated by these diagonal generators, the above framework would easily be able to find the distribution for those graphs as well.

Fourier View

For the second part of this paper, we will consider random walks on groups. To analyze these walks effectively, we need to introduce some algebraic machinery—specifically, the **Fourier transform on groups**.

Let G be a finite group, and let $f : G \to \mathbb{C}$ be a complex-valued function on G. The **Fourier transform** of f at a representation ρ of G is defined by

$$\widehat{f}(\rho) = \sum_{g \in G} f(g)\rho(g),$$

where $\rho : G \to \mathrm{GL}(V) \cong M_{n \times n}(\mathbb{C})$ is a (complex) representation of the group. That is, $\rho(g)$ assigns a matrix to each group element $g \in G$ in a way that respects the group operation: $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$. These matrices often act on a complex vector space V, and their action reflects the structure of the group.

A key concept in this context is that of an **irreducible representation**. A representation ρ is called **irreducible** if there is no nontrivial subspace $W \subset V$ (i.e., $W \neq \{0\}$ and $W \neq V$) such that $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$. Intuitively, this means the representation cannot be decomposed into simpler, smaller representations—it is a "building block" for all representations of G. If the above was true, then we could just consider $p(g)$ as transforming just W instead of all of V. A very important irreducible representation is where $\rho(g) = 1$ for all $g \in G$. This is called the trivial representation and is denoted as ρ_1 in this paper as the first of the irreducible representations of a group. Irreducible representations play a crucial role in Fourier analysis on groups, much like sines and cosines do in classical Fourier analysis on the real line. [3]

An important result in representation theory is that Abelian Groups have 1 dimensional irreducible representations. This is a fact we will use soon. It is also worth mentioning that Abelian groups have as many irreducible representations as they have elements. [3]

We also must introduce the inverse Fourier transform. Let R be the set of irreducible representations of G.

$$f(g) = \frac{1}{|G|} \sum_{\rho \in R} d_\rho Trace(p(g^{-1})\widehat{f}(\rho))$$

Where d_ρ is the dimension of the transformation or the number of the rows of the matrices that ρ maps the elements of g to. On abelian groups, this looks like

$$f(g) = \frac{1}{|G|} \sum_{\rho \in R} p(g^{-1})\widehat{f}(\rho)$$

as $d_\rho = 1$.

One other theorem that is often useful is Plancehrel's theorem which is stated as.

$$\sum_{a \in G} f(a)g(a^{-1}) = \frac{1}{|G|} \sum_{\rho \in R} d_\rho Trace(\widehat{f}(\rho)\widehat{g}(\rho))$$

With abelian groups, we have

$$\sum_{a \in G} f(a)g(a^{-1}) = \frac{1}{|G|} \sum_{\rho \in R} \widehat{f}(\rho)\widehat{g}(\rho)$$

It is also important to note that for a non-trivial representation ($\rho \neq \rho_1$). This is proven in [6]

$$\sum_{g \in G} \rho(g) = 0$$

The most important reason we introduce this machinery into this paper is the fact that Fourier Transform turns convolutions into products. If for all $a \in G$.

$$h(a) = \sum_{s \in G} f(a)g(as^{-1})$$

$$\widehat{h}(\rho) = \widehat{f}(\rho)\widehat{g}(\rho)$$

Expected Values

For each successive time step, we have that for $x, y \in G$. $p^*(x, y)$ is the probability of moving from x to y. For random walks on groups, we are assuming a time-independent increment distribution. Therefore, we can define a new function $p(g) = p^*(x, xg) = P(\tau_{x,xg} = 1)$.

For this portion of the paper, we are concerned with the distribution of $\tau_{g,e}$ on finite abelian groups G.

Much of the ideas of the particular section are from the paper [6]. Before, we approach talking about distributions. We will use Fourier analysis on groups to analyze these functions. Let us begin with functions for expected value of random walks on abelian. Suppose $h(g) = E[\tau_{e,g}]$ where e is the identity of the group.

We have the following recursion relation.

$$E[\tau_{e,g}] = \sum_{s \in G} p(s) E[\tau_{e,gs^{-1}} + 1]$$

Which is expressed as

$$h(g) = 1 + \sum_{s \in G} p(s) h(gs^{-1})$$

It also follows that

$$h(e) = 0$$

Let $\tau_e^+ = \{t \geq 1 | X_0 = e, X_t = e\}$ be the first return time. It is a well known result that $E[\tau_e^+] = \frac{1}{P_s(e)}$ where $P_s(e)$ is the stable probability of being at the e node or the ergodic probability. Since we are fixing our walk such that $P(\tau_{g,h} = 1) = P(\tau_{g_1 g, g_1 h} = 1) = p(g^{-1}h)$ for all $g_1 \in G$, it follows that all nodes have the same stable probability. Therefore, $P_s(e) = \frac{1}{|G|}$. For ease, we will say that $|G| = b$, so we have that We are given that $E[\tau_e^+] = b$. Thus, $h(g)$ satisfies the relation

$$h(g) = k(g) + \sum_{s \in G} p(s)\, h(gs^{-1}),$$

where

$$k(g) = \begin{cases} 1 & \text{if } g \neq e, \\ 1 - b & \text{if } g = e. \end{cases}$$

The adjustment at $g = e$ ensures that $h(e) = 0$; without subtracting n, $h(e)$ would otherwise be equal to b. We will take the fourier transform of k here.

$$\widehat{k}(\rho_j) = \sum_{g \in G} k(g) \rho_j(g)$$

$$\widehat{k}(\rho_j) = \sum_{g \in G} \rho_j(g) - bI$$

Since $\sum_{g \in G} \rho_j(g) = 0$ for $j \neq 1$.

$$\widehat{k}(\rho_j) = -bI$$

So then we have for $j \neq 1$

$$\widehat{h}(\rho_j) = -bI + \widehat{h}(\rho_j)\widehat{p}(p_j)$$

$$\widehat{h}(\rho_j) - \widehat{h}(\rho_j)\widehat{p}(p_j) = -bI$$

$$\widehat{h}(\rho_j)(I - \widehat{p}(p_j)) = -bI$$

$$\widehat{h}(\rho_j) = -b(I - \widehat{p}(p_j))^{-1}$$

We note that by Fourier Inversion and the fact that abelian groups have as many irreducible representations as

$$0 = h(e) = \sum_{j=1}^{b} \widehat{h}(\rho_j)p(e^{-1})$$

$$\sum_{j=1}^{b} \widehat{h}(\rho_j) = 0$$

$$-\sum_{j=2}^{b} \widehat{h}(\rho_j) = \widehat{h}(\rho_1)$$

Since, as we noted above, abelian groups have one dimensional representations.

$$-\sum_{j=2}^{b} \widehat{h}(\rho_j) = \widehat{h}(\rho_1)$$

Plugging this into Fourier inversion, we get

$$h(g) = \frac{1}{b} \sum_{j=1}^{b} \widehat{h}(\rho_j)\rho_j(g^{-1})$$

$$h(g) = \frac{1}{b} \sum_{j=1}^{b} \frac{b}{1 - \widehat{p}(\rho_j)} \rho_j(g^{-1})$$

$$h(g) = \frac{1}{b} \sum_{j=2}^{b} \frac{b}{1 - \widehat{p}(\rho_j)} \rho_j(g^{-1}) + h(\rho_1)$$

$$h(g) = -\sum_{j=2}^{b} \frac{1}{1 - \widehat{p}(\rho_j)} \rho_j(g^{-1}) + \sum_{j=2}^{b} \frac{1}{1 - \widehat{p}(\rho_j)}$$

$$h(g) = \sum_{j=2}^{b} \frac{1 - \rho_j(g^{-1})}{1 - \widehat{p}(\rho_j)}$$

This gives us a formula of first hitting time from e to g in terms of the irreducible representations of G. Do remember that the above formula applies to only Abelian groups.

Variance

Let $q(g) = \mathbb{E}[\tau_{e,g}^2]$ denote the second moment of the first-passage time from the identity element e to a group element $g \in G$.

To derive a recurrence for $q(g)$, we condition on the first step of the random walk:

$$q(g) = \sum_{s \in G} p(s)\, \mathbb{E}[(\tau_{e,gs^{-1}} + 1)^2].$$

Expanding the square inside the expectation:

$$q(g) = \sum_{s \in G} p(s) \left(\mathbb{E}[\tau_{e,gs^{-1}}^2] + 2\mathbb{E}[\tau_{e,gs^{-1}}] + 1 \right).$$

Using $q(gs^{-1}) = \mathbb{E}[\tau_{e,gs^{-1}}^2]$ and $h(gs^{-1}) = \mathbb{E}[\tau_{e,gs^{-1}}]$, we obtain:

$$q(g) = \sum_{s \in G} p(s) q(gs^{-1}) + 2 \sum_{s \in G} p(s) h(gs^{-1}) + 1.$$

We also have the boundary condition:

$$q(e) = 0.$$

To proceed with Fourier analysis, define the adjustment function:

$$k(g) = \begin{cases} 1 & \text{if } g \neq e, \\ 1 - q^* & \text{if } g = e, \end{cases}$$

where $q^* = \mathbb{E}[\tau_e^{+\,2}]$ is the second moment of the return time to e. Unlike the expectation, this quantity depends on the specific structure of the group and transition probabilities.

This allows us to rewrite the recurrence:

$$q(g) = \sum_{s \in G} p(s) q(gs^{-1}) + 2 \sum_{s \in G} p(s) h(gs^{-1}) + k(g).$$

Taking Fourier transforms:

$$\widehat{q}(\rho_j) = \widehat{p}(\rho_j)\widehat{q}(\rho_j) + 2\widehat{p}(\rho_j)\widehat{h}(\rho_j) + \widehat{k}(\rho_j).$$

As in the expectation case, we have:

$$\widehat{k}(\rho_j) = -q^* I.$$

Rearranging the terms:

$$\widehat{q}(\rho_j)(I-\widehat{p}(\rho_j)) = -2\widehat{p}(\rho_j)\widehat{h}(\rho_j) - q^* I.$$

Substituting the earlier expression for $\widehat{h}(\rho_j) = -b(I-\widehat{p}(\rho_j))^{-1}$, we obtain:

$$\widehat{q}(\rho_j) = \left(2b(I-\widehat{p}(\rho_j))^{-1}\widehat{p}(\rho_j) - q^* I\right)(I-\widehat{p}(\rho_j))^{-1}.$$

We will compute using a similar with the inverse as above

$$0 = q(e) = \frac{1}{|G|}\sum_{\rho\in R}\widehat{q}(\rho)\rho(e^{-1})$$

$$0 = \sum_{\rho\in R}\widehat{q}(\rho)$$

As we focus on abelian groups, we have as many abelian groups as elements of our groups

$$\widehat{q}(\rho_1) = -\sum_{j=2}^{b}\widehat{q}(\rho_j)$$

Now, taking the inverse Fourier transform

$$q(g) = \frac{1}{b}\sum_{j=1}^{b}\widehat{q}(\rho_j)\rho_j(g^{-1})$$

Plugging in and taking into account that abelian representations are 1 dimensional

$$q(g) = \frac{1}{b}\sum_{j=2}^{b}\left(\frac{2b\widehat{p}(\rho_j)}{(1-\widehat{p}(\rho_j))^2} - \frac{q^*}{(1-\widehat{p}(\rho_j))}\right) - \frac{1}{b}\sum_{j=2}^{b}\left(\frac{2b\widehat{p}(\rho_j)}{(1-\widehat{p}(\rho_j))^2} - \frac{q^*}{(1-\widehat{p}(\rho_j))}\right)\rho_j(g^{-1})$$

$$q(g) = \frac{1}{b}\sum_{j=2}^{b}\left(\frac{2b\widehat{p}(\rho_j)}{(1-\widehat{p}(\rho_j))^2} - \frac{q^*}{(1-\widehat{p}(\rho_j))}\right)(1-\rho_j(g^{-1}))$$

Therefore, we have the variance

$$Var[\tau_{e,g}] = q(g) - h(g)^2$$

$$= \frac{1}{b}\sum_{j=2}^{b}\left(\frac{2b\widehat{p}(\rho_j)}{(1-\widehat{p}(\rho_j))^2} - \frac{q^*}{(1-\widehat{p}(\rho_j))}\right)(1-\rho_j(g^{-1})) - \left(\sum_{j=2}^{b}\frac{1-\rho_j(g^{-1})}{1-\widehat{p}(\rho_j)}\right)^2$$

Distributions

For the ease of writing, we will define that $P(\tau_{g,e} = n) = m_n(g)$. It follows that $p(g) = m_1(g)$ It follows that we have

$$m_n(g) = \sum_{s \in G} p^*(g, gs^{-1}) m_{n-1}(gs^{-1})$$

and

$$m_n(e) = 0$$

for $n \geq 1$. So, we can design a new function c_n

$$c_n(g) = \begin{cases} 0 & g \neq e \\ \sum_{s \in G} m_{n-1}(s) m_1(s) & g = e \end{cases}$$

$$m_n(g) = -c_n(g) + \sum_{s \in G} m_1(s^{-1}) m_{n-1}(gs^{-1})$$

For the purposes of this write-up, we will assume that our random walk is symmetric.

$$m_n(g) = -c_n(g) + \sum_{s \in G} m_1(s) m_{n-1}(gs^{-1})$$

$$\widehat{m_n}(\rho_j) = -I \sum_{s \in G} m_{n-1}(s) p(s) + \widehat{p}(\rho_j) \widehat{m_{n-1}}(\rho_j)$$

$$\widehat{m_n}(\rho_j) = -I \sum_{s \in G} m_{n-1}(s) m_1(s^{-1}) + \widehat{m_1}(\rho_j) \widehat{m_{n-1}}(\rho_j)$$

By Plancharel's Theorem on Abelian Groups and the fact that Abelain groups have exactly the same number of irreducible representations and group elements.

$$\widehat{m_n}(\rho_j) = -\frac{1}{k} \sum_{a=0}^{k-1} \widehat{m_{n-1}}(\rho_a) \widehat{m_1}(\rho_a) + \widehat{m_1}(\rho_j) \widehat{m_{n-1}}(\rho_j)$$

and so then we have that

$$\widehat{m_n}(\rho_j) = -\frac{1}{k} \sum_{a=0}^{k-1} \widehat{m_{n-1}}(\rho_a) \widehat{m_1}(\rho_a) + \widehat{m_1}(\rho_j) \widehat{m_{n-1}}(\rho_j)$$

So then we have a recurrence relation such that.

$$A \begin{bmatrix} \widehat{m_{n-1}(\rho_0)} \\ \widehat{m_{n-1}(\rho_1)} \\ \dots \\ \widehat{m_{n-1}(\rho_{k-1})} \end{bmatrix} = \begin{bmatrix} \widehat{m_n(\rho_0)} \\ \widehat{m_n(\rho_1)} \\ \dots \\ \widehat{m_n(\rho_{k-1})} \end{bmatrix}$$

Where

$$A = \begin{bmatrix} \frac{k-1}{k}\widehat{m_1}(\rho_0) & -\frac{1}{k}\widehat{m_1}(\rho_1) & ... & -\frac{1}{k}\widehat{m_1}(\rho_{k-1}) \\ -\frac{1}{k}\widehat{m_1}(\rho_0) & \frac{k-1}{k}\widehat{m_1}(\rho_1) & .... & -\frac{1}{k}\widehat{m_1}(\rho_{k-1}) \\ ... & ... & ... & ... \\ -\frac{1}{k}\widehat{m_1}(\rho_0) & -\frac{1}{k}\widehat{m_1}(\rho_1) & .... & \frac{k-1}{k}\widehat{m_1}(\rho_{k-1}) \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{k-1}{k} & -\frac{1}{k} & ... & -\frac{1}{k} \\ -\frac{1}{k} & \frac{k-1}{k} & .... & -\frac{1}{k} \\ ... & ... & ... & ... \\ -\frac{1}{k} & -\frac{1}{k} & .... & \frac{k-1}{k} \end{bmatrix} \begin{bmatrix} \widehat{p}(\rho_0) & 0 & ... & 0 \\ 0 & \widehat{p}(\rho_1) & .... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & .... & \widehat{p}(\rho_{k-1}) \end{bmatrix}$$

Which means that we have that

$$A^{n-1}\widehat{m_1} = \widehat{m_n}$$

I haven't made any assumptions so far except for symmetric random walks and abelian groups. The above matrix is similar to Q so it has the same eigenvalues. Taking the powers of the above matrix is faster and it is generally easier to compute the eigenvalues of such a matrix. For large matrices, the above's eigenvalues and eigenvectors will faster to compute due to the fact that it is a diagonal matrix multiplied by a nilpotent matrix.

Spectral Methods

Before we introduce the last part of this paper, we must discuss some basic lemmas. The first is that G is d-regular, then λ_1 or the largest eigenvalue is equal to d. It's also true that $|\lambda_n| \leq d$ as well.

Another lemma is that Trace(A^k) counts the number of closed k length closed walks in the graph.

$$\sum_{i=1}^{V} \lambda_i^k = \text{Trace}(A^k)$$

The next part of the paper will also only work on **Vertex-Transitive Graphs**. Vertex Transitive Graphs are graphs where there exists a graph automorphism ϕ such that $\phi(i) = j$ for any vertices i and j. In simple terms, it means the graph looks the same from every vertex. It also means that each node has the same amount of closed walks length k that start and end at that node. This quantity is $\frac{\text{Trace}(A^k)}{V}$ where V is the number of vertices in the graph. [2]

Loop Counting

Let $M_{n_{i,j}} = P(\tau_{i,j} = n)$. It is clear that any vertex transitive graph that we have.

$$M_0 = I$$

and

$$M_1 = \frac{1}{d}A^1 - \frac{1}{d}\frac{Trace(A)}{V}I$$

This makes sure that a vertex can't visit itself in one move. We also have.

$$M_2 = \frac{1}{d^2}A^2 - \frac{1}{d}\frac{Trace(A)}{V}M_1 - \frac{1}{d^2}\frac{Trace(A^2)}{V}M_0$$

This has it so we can't visit our selves in 2 moves or 1 move. Generalizing this

$$M_n = \frac{1}{d^n}A^n - \sum_{k=1}^{n}\frac{1}{d^k}\frac{Trace(A^k)}{V}M_{n-k}$$

Basically this says, we are not allowed to visit the node earlier than n moves and then make a loop to that same node to achieve a walk of technically *technically* n moves from i to j. We get this sum.

$$\sum_{k=0}^{n}\frac{1}{d^k}\frac{Trace(A^k)}{V}M_{n-k} = \frac{1}{d^n}A^n$$

As this is a Cauchy product, we get the following

$$\sum_{n=0}^{\infty}\frac{t^n}{d^n}\frac{Trace(A^n)}{V}\sum_{n=0}^{\infty}M_n t^n = \sum_{n=0}^{\infty}\frac{t^n}{d^n}A^n$$

By definition of trace we have.

$$\frac{1}{V}\left(\sum_{n=0}^{\infty}\frac{t^k}{d^k}\sum_{i=0}^{V}\lambda_i^k\right)\sum_{n=0}^{\infty}M_n t^n = \sum_{n=0}^{\infty}\frac{t^n}{d^n}A^n$$

$$\frac{1}{V}\left(\sum_{i=0}^{V}\sum_{n=0}^{\infty}\lambda_i^k\frac{t^k}{d^k}\right)\sum_{n=0}^{\infty}M_n t^n = \sum_{n=0}^{\infty}\frac{t^n}{d^n}A^n$$

$$\frac{1}{V}\left(\sum_{i=0}^{V}\frac{1}{1-\frac{\lambda_i t}{d}}\right)\sum_{n=0}^{\infty}M_n t^n = (I - \frac{t}{d}A)^{-1}$$

so for $|t| < 1$

$$\sum_{n=0}^{\infty}M_n t^n = \frac{V(I-\frac{t}{d}A)^{-1}}{\sum_{i=0}^{V}\frac{1}{1-\frac{\lambda_i t}{d}}}$$

$$\sum_{n=1}^{\infty}e_i^T M_n t^n e_j = \frac{V\frac{\det_{j,i}(I-\frac{t}{d}A)}{\prod_{i=1}^{V}(1-\frac{t}{d}\lambda_i)}}{\frac{\sum_{j=1}^{V}\prod_{i\neq j}(1-\frac{t}{d}\lambda_i)}{\prod_{i=1}^{V}(1-\frac{t}{d}\lambda_i)}}$$

$$\sum_{n=1}^{\infty}P(\tau_{i,j}=n)t^n = \frac{V\det_{j,i}(I-\frac{t}{d}A)}{\sum_{j=1}^{V}\prod_{i\neq j}(1-\frac{t}{d}\lambda_i)}$$

This is a very general result for all vertex transitive graphs. Which apply to all Cayley graphs as well.

Small Values of M_n

To get an idea of how the values of M_n relate to A, we can see how they are related.

$$
\begin{aligned}
M_0 &= I \\
M_1 &= \frac{1}{d}A - \frac{1}{d}\cdot\frac{\mathrm{Tr}(A)}{V}I \\
M_2 &= \frac{1}{d^2}\left(A^2 - \frac{\mathrm{Tr}(A)}{V}A + \frac{\mathrm{Tr}(A^2) - V\mathrm{Tr}(A)}{V^2}I\right) \\
M_3 &= \frac{1}{d^3}\left(A^3 - \frac{\mathrm{Tr}(A)}{V}A^2 + \frac{\mathrm{Tr}(A)^2 - V\mathrm{Tr}(A^2)}{V^2}A\right. \\
&\quad \left. - \frac{\mathrm{Tr}(A)^3 - 2V\mathrm{Tr}(A^2)\mathrm{Tr}(A) + V^2\mathrm{Tr}(A^3)}{V^3}I\right) \\
M_4 &= \frac{1}{d^4}\left(A^4 - \frac{\mathrm{Tr}(A)}{V}A^3 + \frac{\mathrm{Tr}(A)^2 - V\mathrm{Tr}(A^2)}{V^2}A^2\right. \\
&\quad - \frac{\mathrm{Tr}(A)^3 - 2V\mathrm{Tr}(A^2)\mathrm{Tr}(A) + V^2\mathrm{Tr}(A^3)}{V^3}A \\
&\quad \left. + \frac{\mathrm{Tr}(A)^4 - 3V\mathrm{Tr}(A^2)\mathrm{Tr}(A)^2 + V^2(2\mathrm{Tr}(A^3)\mathrm{Tr}(A) + \mathrm{Tr}(A^2)^2) - V^3\mathrm{Tr}(A^4)}{V^4}I\right)
\end{aligned}
$$

After M_4, it becomes very hard to write the full sum on one page. There does seem to be a pattern where $M_n = \frac{1}{d}AM_{n-1} - f(A)I$ where $f(A)$ is made up of partitions of n expressed by the trace split up by how many ways there are to sum to that number. Future work would prove this exactly.

Sample Distributions

Using Mathematica, we can code up the series

$$\sum_{n=1}^{\infty} P(\tau_{i,j} = n)t^n = \frac{V\det_{j,i}(I - \frac{t}{d}A)}{\sum_{j=1}^{V}\prod_{i\neq j}(1 - \frac{t}{d}\lambda_i)}$$

to get series that describe our distributons.

- 3d Hypercube Walk from $i = (0,0,0) \to j = (1,1,1)$

$$\sum_{n=0}^{\infty} P(\tau_{i,j} = n)t^n = \frac{2t^3}{9} + \frac{14t^5}{81} + \frac{98t^7}{729} + \frac{686t^9}{6561} + \frac{4802t^{11}}{59049} + O\left(t^{13}\right)$$

- Walk in S_3 from $i = e$ to $j = (1,3)$

$$\sum_{n=0}^{\infty} P(\tau_{i,j} = n)t^n = \frac{t}{3} + \frac{4t^3}{27} + \frac{2t^4}{27} + \frac{20t^5}{243} + \frac{44t^6}{729} + \frac{116t^7}{2187} + \frac{280t^8}{6561} + O(t^9)$$

- Walk in D_8 from $i = e$ to $j = (14)(23)$

$$\sum_{n=0}^{\infty} P(\tau_{i,j} = n)t^n = \frac{t}{3} + \frac{4t^3}{27} + \frac{28t^5}{243} + \frac{196t^7}{2187} + \frac{1372t^9}{19683} + \frac{9604t^{11}}{177147} + O\left(t^{13}\right)$$

It is important to note that the generators of S_3 in this example are the transpositions and that generators of D_8 are (14)(23) and (1234) and (12)(34). As above, we walk along our generators with equal probability.

Continuous-Time Random Walks

We consider a continuous-time random walk on a finite state space, where transitions between states occur according to a Poisson process with rate $\lambda = 1$. Let M be the markov matrix defining this walk. Let j be the defined as the end point. Let Q be the substochastic matrix of M with the jth row and column removed. Let P_1 again be the vector with the 1 step probabilities.

Define $\tau^c_{*,j}$ as the vector random variable representing the first hitting times to node j from every other node. We are interested in computing the cumulative distribution function (CDF) of $\tau^c_{*,j}$, that is,

$$P(\tau^c_{*,j} \leq t).$$

In continuous time, the probability that the walk makes exactly n transitions by time t is given by the Poisson distribution:

$$P(\text{makes } n \text{ steps}) = \frac{t^n e^{-t}}{n!}.$$

Therefore, the probability that the walk hits node j within time t can be written as

$$P(\tau^c_{*,j} \leq t) = \sum_{n=0}^{\infty} \frac{t^n e^{-t}}{n!} \cdot P(\text{hits } j \text{ in at most } n \text{ steps}).$$

The second term in the sum can be written as

$$\sum_{k=1}^{n} Q^{k-1} P_1,$$

Hence,

$$P(\tau^c_{*,j} \leq t) = \sum_{n=0}^{\infty} \frac{t^n e^{-t}}{n!} \sum_{k=1}^{n} Q^{k-1} P_1.$$

We can interchange the order of summation:

$$P(\tau^c_{*,j} \leq t) = \sum_{k=1}^{\infty} Q^{k-1} P_1 \sum_{n=k}^{\infty} \frac{t^n e^{-t}}{n!}.$$

Recognizing that this is the tail of the Poisson distribution, we move to a matrix formulation. Using the identity

$$\sum_{n=0}^{\infty} \frac{t^n e^{-t}}{n!} Q^n = e^{-t(I-Q)},$$

we obtain a closed-form expression:

$$P(\tau^c_{*,j} \leq t) = (I-Q)^{-1} \left(I - e^{-t(I-Q)}\right) P_1. \text{ for } t > 0$$

Differentiating this expression with respect to t, we obtain the vector probability density function (PDF):

$$f_{\tau^c_{*,j}}(t) = \frac{d}{dt} P(\tau^c_{*,j} \leq t) = e^{-t(I-Q)} P_1.$$

We can interpet this as smoothed out version of our orginial walk. This can make the moments easier to obtain as the above distribution has the same moments as our discrete time random walk.

$$\int_0^{\infty} t e^{-t(I-Q)} P_1 = (I-Q)^{-2} P_1 = \sum_{n=0}^{\infty} n Q^{n-1} P_1$$

$$\int_0^{\infty} t^2 e^{-t(I-Q)} P_1 = 2(I-Q)^{-3} P_1 = \sum_{n=0}^{\infty} n^2 Q^{n-1} P_1$$

And so on.

Future Work

There is a lot future work in this field. Finding q^* values for abelian groups would help variance calculations become tractable. It would also be an advance in the field for general graphs as second or higher moments of return times are not generally well studied.

To obtain distributions from M_n we need to extract coefficients using residue integrals. This calculation will be in the eventual our paper written on our findings.

Another direction one could take is by calculating how much information one would need to have about the M_n matrices to get the adjacency matrix of the graph. It is clear that recovering the matrix from the distribution of one pair of vertices is not sufficient (consider cycle and path graphs). This begs the question of how many distributions are needed. Since we have that A^k are in the Bose-Mesner Algebra by the following recursion explored earlier,

$$\sum_{k=0}^{n} \frac{1}{d^k} \frac{Trace(A^k)}{V} M_{n-k} = \frac{1}{d^n} A^n$$

we can retrieve A from M_n. Then, therefore there exists some proportion of M_n that is needed before extracting A.

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