Review of

The Erdős Distance Problem [3]

by Julia Garibaldi, Alex Iosevich, Steven Senger

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1 Introduction

(All of the papers mentioned in this review are available at www.cs.umd.edu/~gasarch/erdos_dist/erdos_dist.html)

The following problem was posed by Erdős in 1946: Given n points in the plane how many distinct distances are there guaranteed to be? We denote this $h(n)$. Erdős showed that $\Omega(\sqrt{n}) \le h(n) \le O(\frac{n}{\sqrt{\log n}})$. The lower bound could be on a High School Math competition. The upper bound is from the $\sqrt{n} \times \sqrt{n}$ grid and requires some number theory to prove. Moser and Chung and this book attribute to Erdős the conjecture that $(\forall \epsilon)[h_n \ge n^{1-\epsilon}]$. This conjecture is not in Erdős's paper; however, it is likely he gave it in lectures.

There has been considerable progress on this problem over the years. Some of the progress has come from geometry and some from pure combinatorics. This book gives proofs of many improvements to the bound on $h(n)$. This book also has material on other metrics, the problem in d-dimensional space, and the problem over finite fields.

Has the problem been solved? The last result on the Erdős Distance Problem in this book is $h(n) \ge n^{\frac{48-14e}{55-16e}}$ which is approximately $n^{0.864137}$. The book then goes on to show why the methods used to obtain that result cannot be extended. Right before the book went to press Katz and Guth showed $h(n) \ge \Omega(\frac{n}{\log n})$, which solved the conjecture (though note that we still have the gap $\Omega(\frac{n}{\log n}) \le h(n) \le O(\frac{n}{\sqrt{\log n}})$). This should makes this book *more interesting* to people in TCS. In TCS we have proven results like "such-and-such technique won't suffice to crack P vs NP" and take this as evidence that P vs NP is hard. For the Erdős distance problem they also proved that "such and such techniques won't suffice" but they *cracked the problem anyway!* To be fair, P vs NP seems much harder than the Erdős Distance Problem; however, it still makes one pause. More to the point, having an example of a barrier result, and then how it was overcome, is interesting. Hence I recommend the reader read this book *and* then go read the Katz-Guth paper.

Moreover, the mathematics in this book is often of independent interest. For example, the Szemerédi-Trotter theorem on incidences in the plane and the crossing lemma for graphs are proven and used in this book; however they have many other applications.

2 Summary of Contents

The first two chapters introduce the problem, give Erdős's proof that $h(n) \ge \Omega(\sqrt{n})$, and Moser's proof that $h(n) \ge \Omega(n^{2/3})$. These proofs are both geometric. The third chapter proves the Cauchy-Schwartz inequality and gives some applications.

The fourth chapter proves the following two theorems that is uses as Lemmas.

1. The Crossing Lemma for graphs: If a graph has n vertices and $e \geq 4n$ edges then the number of crossings is at least $\Omega(\frac{e^3}{n^2})$.
2. The Crossing Lemma for multi-graphs: If a multigraph has multiplicity $\leq m$, n vertices and $e \geq 5mn$ edges then the number of crossings is at least $\Omega(\frac{e^3}{mn^2})$.
3. The Szemerédi-Trotter theorem on incidences in the plane: For an set of P points and L lines in the plane the number of incidences of points on lines is at most $O(P + L + (LP)^{2/3})$.

These are both used in the fifth chapter to obtain better lower bounds on $h(n)$. Just using the crossing lemma for multigraphs you can obtain $h(n) \geq \Omega(n^{2/3})$ approximately $\Omega(n^{0.67})$. which, alas, we already have. But if you also use the Szemerédi-Trotter theorem then one can obtain $\Omega(n^{0.8})$. This is a result of Székely. This is a very nice proof since you translate the problem to one in pure combinatorics and then solve it there with the crossing lemma.

The sixth chapter proves $h(n) \geq \Omega(n^{6/7})$ (note that $6/7$ is approximately 0.8574). This is a careful argument involving taking, for each point p, THREE points that p_1, p_2, p_3 such that $|p - p_1| = |p - p_2| = |p - p_3|$. The seventh chapter extends the argument to FIVE points. More than that, this chapter casts the $n^{6/7}$ argument in a new light (same proof, different way of looking at it) so that one sees how you could *try* to generalize it. The better results depend on theorems from pure combinatorics. Here is the key combinatorial question they tackle: Given k find a small α_k such that the following holds: for all an $M \times k$ matrices of distinct elements, if S is the set of pairwise distinct sums of entries of A in the same row, then $M \leq O(S^{\alpha_k})$. Gabor Tardos obtained, for $k = 5$, $\alpha = 11/4$ which yields $h(n) \geq \Omega(n^{44/51})$ (approximately $n^{0.8627}$, using sets of FIVE points. Katz and Tardos later obtained $k = 5$, $\alpha = 19/7$ which yields $h(n) \geq \Omega(n^{19/22})$ (approximately $n^{0.8636}$. The best result using these techniques (and higher values of k), also due to Katz and Tardos, is $h(n) \geq n^{48-14e}55 - 16e$ which is approximately $n^{0.864137}$ (the proof of this is not presented). The book then gives a proof by Imre Ruzsa that, using these techniques, this result is optimal.

The seventh chapter is about information theory. What does information theory have to do with the Erdős distance problem? Using information theory one can obtain results like the ones above about matrices.

3 Opinion

Hypothetically anyone could read this book. Virtually all of the math that you need is in it. But of course there is the issue of *Mathematical Maturity*. A good Junior Math major should be able to read and understand most of the book, though some parts will be tough going. The authors leave many of the results for the exercises. This makes the book harder to read but this does force you to be involved.

Is the Erdős distance problem a good problem? Yes. Hilbert said that a good math problem *should be difficult in order to entice us, yet not completely inaccessible lest it mock our efforts*. The interesting mathematics that has been applied to it, and come out of, make it enticing. The steady progress shows that the problem does not Mock our efforts some do (Collatz Conjecture- I'm talking about you!)

Is this the book to read on the problem? The sarcastic (and unfair) answer is *yes, because its the only one*. However, in absolute terms this is a good book on the problem and will take the reader through much math of interest.