Basic skills II: summation by parts, dyadic blocks and infinite sums

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From finite to infinite

- In the previous lecture, we considered geometric series and obtained the basic formula

$$A^k + A^{k+1} + \cdots + A^n = \frac{A^{n+1} - A^k}{A - 1}; \text{ if } A \neq 1.$$

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- The first step is to understand what it means to sum an infinite number of terms.

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- $$\text{if } \lim_{N \to \infty} S_N \text{ exists.}$$

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$$\lim_{N\to\infty} S_N = L$$
- if given $\epsilon > 0$ there exists $M > 0$ such that
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- Suppose that
$$S_N = \frac{N+1}{N} = 1 + \frac{1}{N}.$$

- When N gets larger and larger, $\frac{1}{N}$ gets smaller and smaller, so we might guess that
$$\lim_{N\to\infty} 1 + \frac{1}{N} = 1.$$

Examples of limits (continued)

- To make this idea precise, we must show that given $\epsilon > 0$ there exists $M > 0$ such that

$$\left|1 + \frac{1}{N} - 1\right| < \epsilon \text{ whenever } N \geq M.$$

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- It is not difficult to see that choosing $M > \frac{1}{\epsilon}$ does the job.

Examples of limits-a harder example

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- and we ask, what is

$$\lim_{N\to\infty} S_N?$$

- We start by observing that

$$\frac{N^2}{N^2+N+1} = \frac{N^2+N+1}{N^2+N+1} - \frac{N+1}{N^2+N+1}.$$

A harder example continued

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- Observe that $N^2 + N + 1 > N^2 + N$, so
$$\left| \frac{N+1}{N^2+N+1} \right| \leq \frac{N+1}{N^2+N} = \frac{1}{N}.$$

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- We now consider a more complicated example that we shall need later in the lecture. Let
$$S_N = \frac{N}{2^N}.$$

- We probably have an intuition that 2^N grows much faster than N, so the limit should be 0, but how do we prove this rigorously?

Subsets of a set of size N

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- There are N subsets of size 1, namely the sets $\{O_1\}$, $\{O_2\}, \ldots, \{O_N\}$.

- How many subsets of size 2 are there? Well, there are N choices for the first element of the set and $N-1$ choices for the second. The order of the elements does not matter, so the number of choices is

$$\frac{N(N-1)}{2}.$$

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$$100010100\ldots 0.$$

- We put a 1 in the k'th slot if O_k is contained in the subset, and 0 otherwise.
- It follows that the total number of subsets is equal to a number of strings of 1's and 0"s of length N. The number of such strings is 2^N since we have two choices for each slot.

Subsets of a set of size N (concluded)

- We just saw that the number of subsets of size 2 is equal to $\frac{N(N-1)}{2}$, and the total number of subsets is 2^N, from which we conclude that

$$\frac{N(N-1)}{2} < 2^N.$$

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- It follows that

$$\frac{N}{2^N} \leq \frac{N}{\frac{N(N-1)}{2}} = \frac{2}{N-1}.$$

Back to $\lim_{N\to\infty} \frac{N}{2^N}$

- We must show that given $\epsilon > 0$, there exists $M > 0$ such that

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- Choose $M > \frac{2}{\epsilon} + 1$. Then

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- Enough of limits for now and back to sums!

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- We have

$$\lim_{N\to\infty} \sum_{k=1}^{N} A^k = \lim_{N\to\infty} \frac{A^{N+1} - A}{A-1} = \lim_{N\to\infty} \frac{A^{N+1}}{A-1} - \frac{A}{A-1}$$

Infinite geometric series (continued)

- $$= \frac{A}{1-A} + \frac{1}{A-1} \lim_{N \to \infty} A^N.$$

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by a slight modification of the arguments we went over.

- If $|A| > 1$, $|A^N| = |A|^N$ is arbitrarily large as N grows, so

$$\lim_{N\to\infty} A^N \text{ does not exist.}$$

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- The limit as $N \to \infty$ of S_N does not exist, but proving this requires some care.

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- Let $\epsilon = \frac{1}{2}$. Then no matter how large N is, either

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- because

$$1 = |S_N - S_{N+1}| = |S_N - L + L - S_{N+1}| \leq |S_N - L| + |L - S_{N+1}|.$$

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- Moreover, we have shown that if $|A| < 1$,

$$\sum_{k=1}^{\infty} A^k = \lim_{N\to\infty} \frac{A^N}{A-1} - \frac{A}{A-1} = \frac{A}{1-A}.$$

Close friends and relatives of the geometric series

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- We shall now consider these sums as $N \to \infty$ and we shall do all the calculations from scratch.

- Our first observation is that there is no point considering the case $|A| \geq 1$ because they will diverge just as in the case of the regular geometric series.

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$\sum_{k=1}^{\infty} kA^k$: taking limits

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- We must now compute

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- We have already seen that $II = 0$ since $|A| < 1$. There is nothing to be done with III, so matters have been reduced to considering I.

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- There are many ways to compute the limit under consideration, but we shall do it by modifying the argument for the case $A = \frac{1}{2}$.

$\lim_{N\to\infty} NA^N$ continued

- We start by observing that if $0 < A < 1$,

$$A^N = 2^{N\log_2(A)} = 2^{-N\log_2(A^{-1})},$$

where $\log_2(A^{-1}) > 0$.

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amounts to showing that if $0 < A < 1$,

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$\lim_{N\to\infty} NA^N$: reduction to counting

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may not be enough because this implies that

- $$2^{N\log_2(A^{-1})} > \left(\frac{N(N-1)}{2}\right)^{\log_2(A^{-1})},$$

- and
$$\left(\frac{N(N-1)}{2}\right)^{\log_2(A^{-1})} \leq N \text{ if } \log_2(A^{-1}) \leq \frac{1}{2}.$$

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- In order to find the way out of this predicament, recall that we concluded that
$$\frac{N(N-1)}{2} < 2^N$$
because the number of subset of size **two** is smaller than the **total** number of subsets of a set consisting of N elements.

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- In order to find the way out of this predicament, recall that we concluded that
$$\frac{N(N-1)}{2} < 2^N$$
because the number of subset of size **two** is smaller than the **total** number of subsets of a set consisting of N elements.

- But the number of subset of size K is smaller than the **total** number of subsets of a set consisting of N elements for any $K \leq N$!

- We conclude that

$$\frac{N!}{(N-K)!K!} \leq 2^N \text{ for } 1 \leq K \leq N.$$

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- Observe that

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- Letting $\log_2(A^{-1}) = \beta$, we conclude that

$$2^{N\beta} \geq \frac{N^{K\beta}}{K^{K\beta}}.$$

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- Then

$$\frac{N^{K\beta}}{K^{K\beta}} \geq \frac{N^3}{K^4},$$

- and this quantity is

$$\geq N^2 \text{ if } N \geq K^4.$$

$\lim_{N\to\infty} NA^N$: conclusion

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- It follows that

$$NA^N = \frac{N}{2^{N\log_2(A^{-1})}} \leq \frac{1}{N},$$

- and we conclude in the same way as before that

$$\lim_{N\to\infty} NA^N = 0.$$

Back to $\sum_{k=1}^{\infty} kA^k$

- We showed above that

$$\sum_{k=1}^{\infty} kA^k = \lim_{N\to\infty} \frac{NA^{N+1}}{A-1} - \lim_{N\to\infty} \frac{A^{N+1}}{(A-1)^2} + \frac{A}{(A-1)^2}.$$

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- We can now conclude that the right hand side is equal to

$$\frac{A}{(A-1)^2}.$$

An example

- In the case $A = \frac{1}{2}$, we see that

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- In one of the subsequent lectures, we are going to show that this sum represents the "expected" number of flips of a fair coin needed to produce heads. Even without knowing much about probability, one might guess that the answer is 2 since the probability of getting heads on the first flip is equal to $\frac{1}{2}$.

Estimating infinite sums

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Harmonic series

- As before, we consider

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- We are going to show that

$$S_{2^m} \geq \frac{m+2}{2}, \text{ which will do the trick.}$$

Dyadic blocks enter the picture

- We have
$$\sum_{k=2^m+1}^{2^{m+1}} \frac{1}{k} \geq \frac{1}{2^{m+1}} \cdot 2^m = \frac{1}{2},$$
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- It follows that

$$S_{2^m} = S_1 + \sum_{j=0}^{m-1} S_{2^{j+1}} - S_{2^j}$$

$$\geq 1 + \frac{m}{2} = \frac{m+2}{2}.$$

Harmonic series concluded

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- But this is blatantly untrue since we have shown that

$$S_{2^m} \geq \frac{m+2}{2}.$$

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$$S_{2^{m+1}} - S_{2^m} = \sum_{k=2^m+1}^{2^{m+1}} \frac{1}{k^2} \leq \frac{1}{2^{2m}} \cdot 2^m$$

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- It follows that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \leq \sum_{m=0}^{\infty} 2^{-m} = 2$$

and we already know that this sum converges.

Some concluding thoughts

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- Much beautiful mathematics lies ahead!!