Review[11] of
A View from the Top
Analysis, Combinatorics and Number Theory
by Alex Iosevich
AMS, 2007
xiii+136 pages, softcover, $30.00

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1 Introduction

This is an undergraduate-level math textbook. It is quite special because instead of specializing on a given topic, it wanders through fields as different as Calculus, Number Theory, Probability, Algebra and Geometry. The mathematics presented in it have no immediate relation to computing and algorithms, except maybe for the fact that one of the chapters is dedicated to Number theory, Probabilities and the Riemann ζ function.

2 Summary

Chapters 1–4

This book operates like a Mahler Symphony: it starts very softly, and before you realize it, you are in deep water, with drums and trumpets sounding all around you. Chapter 1 deals with Cauchy-Schwartz[12] inequality in its simplest form $\sum a_k b_k \leq \sqrt{\sum a_k^2}\sqrt{\sum b_k^2}$, proven by just a few lines of "high school math" (a much simpler way to express it is $\langle a, b\rangle \leq \|a\| \cdot \|b\|$).

This inequality is true, but the author can do better, because, as he says, "to say something interesting, one must walk on the very edge of the cliff of falsehood, yet never fall off." He gives a better version of it, the Hlder inequality: $\sum a_k b_k \leq (\sum a_k^m)^{\frac{1}{m}}(\sum b_k^{m'})^{\frac{1}{m'}}$, where m' is the "dual" of m given by $\frac{1}{m} + \frac{1}{m'} = 1$. There is also a generalization of this formula to n-term products.

In Chapter 2 the author deals with a first unexpected use of Cauchy-Schwartz inequality: take N distinct points in $\mathbb{R}^3$ and their canonical projections to the three planes, and *count the number of distinct points you obtain on the planes.* We want that number to be minimal. If you think about it, one can always choose the points in such a way that their projection on *one* plane is minimal: a single point, for example. But once you have done it for one plane, it is hard, or even impossible to reduce the number of projected points on the other planes.

Using Cauchy-Schwartz, the author quantifies this phenomenon: if S_N is the set of points, and $\#\pi_j(S_N)$ the cardinal of the j-th projection, then, however you may arrange the points in space, at least one of these numbers will always be greater than $\sqrt[3]{N^2}$ (for example, for 100 points, you will always have at least 22 points projected on one of the planes).

[12] Amusingly enough, the author breaks a record by writing the same name in three different ways on a single page (p. 4): Schwarz (the German way), Schwartz (the French way) and Schwarts (the Yiddish way).

In Chapter 3 he deals with two-dimensional projections of a set of points in $\mathbb{R}^4$. The result is that the cardinal of the set is upper bounded by the cubic root of the product of cardinals of projections on planes $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$ and $(3,4)$. To prove this he uses a very interesting interpolation result which goes as follows: let $\|a_*\|_s$ be $(\sum a_*^s)^{\frac{1}{s}}$, then if $\|a_*\|_1 \leq C_1$ and $\|a_*\|_2 \leq C_2$ then for an arbitrary $1 < s < 2$ we have $\|a_*\|_s \leq C_1^{\frac{2}{s}-1} C_2^{2-\frac{2}{s}}$.

The general case of k-dimensional projections if $\mathbb{R}^d$ has been treated by Loomis-Whitney in [4].

Chapter 4 is again about projections, but this time from a different point of view. It deals with comparisons of unit balls (balls of volume 1) and unit cubes in $\mathbb{R}^d$. For example, take the 2-ball (that's a circle). The 2-cube (that's a square) does not fit inside the 2-ball, but the projection of the 2-ball (a centered segment of length $\frac{2}{\sqrt{\pi}}$) contains the projection of the 1-cube (if this projection is parallel to a side of the square). The author asks the question: what should be the dimension of our projections for this to happen? It turns out that for an n-ball, we need to project onto a subspace of dimension $k \leq \frac{4}{\pi}(n!)^{\frac{1}{n}}$ (for $n = 2$ we do find $k = 1$). Having established that, the author delves into approximations of this formula, leading to the well-known Stirling formula.

Chapter 5

This is where we get into deep water. Given is the *incidence problem*: what is the maximum number of intersections of n lines? By simply using the Cauchy-Schwartz inequality, the author shows that it is less than $\sqrt{2}n^{\frac{3}{2}}$. He states the "sharp answer" (given in [5]): it is less than $Cn^{\frac{4}{3}}$, where C is a positive constant.

From this, the author derives quite a surprising result: take a complete graph with N vertices embedded into $\mathbb{R}^2$, and measure the lengths of edges. We are interested in the minimum number of such lengths. One would expect that using symmetry it would be easy to calculate the minimum number of distinct edge lengths of a complete planar graph.

Well, this is not the case. Using the results of the book, one can prove that the number of distinct edge lengths is larger than $C\sqrt{N}$, and the author states that the best known lower bound is CN^{β} with $\beta \approx .86$ (cf. [3]) and that there is a conjecture on the value $C\frac{N}{\sqrt{\log(N)}}$. Like Goldbach, this is yet another conjecture which is so easy to formulate that one can only wonder how come it hasn't been proved yet.

And here the author brilliantly jumps into a different domain of math: number theory. Take A to be a finite subset of $\mathbb{Z}$ and count the number of all possible sums $a + a'$ and products $a \cdot a'$, where $a, a' \in A$. If $\#A$ is the cardinal of A then the cardinal of such sums and products is lower bounded by $C(\#A)^{\frac{5}{4}}$.

And then, switching to yet another domain, namely graph theory, the author proves that the number of crossings of a planar graph G with n vertices and e edges, is minorated by $C\frac{e^3}{n^2}$.

If you think this is as acrobatic as it can get, you are wrong: to prove this result the author uses probability methods (!). The authors of [5] take a random subgraph G' of G where every vertex is chosen with uniform probability p: then the expectation of "surviving" vertices is np, the one of edges is ep^2, and the expected value of the crossing number is upper bounded by p^4 times the crossing number of G.

Chapters 6–8

After calculus, graph theory, number theory and probability methods, we turn now to algebra, more precisely to vector spaces over finite fields $\mathbb{F}_p$ (where p is prime). After explaining why $\mathbb{F}_p$ is a field and how we can define vector spaces and lines on it, the author presents the Besicovitch-Kakeya conjecture, which states the following: if B is a subset of a d-dimensional vector space over $\mathbb{F}_p$, big enough to contain, for a given point x, lines going through x with all possible slopes, then the size of B is lower bounded by Cp^d, where C is a positive constant. We call a point of intersection of several lines a "bush," and the conjecture says that for every such vector field there is a large bush.

Chapter 7 proves this conjecture for R^2 using (guess what!) the Cauchy-Schwartz inequality [1]. And Chapter 8 develops some of the difficulties of the high-dimensional case of the conjecture.

Chapters 9–10

After a long, although elementary in nature, introduction to probabilities (Chapter 9) the author deals with the interaction of probabilities and number theory: how can we estimate the probability that two (arbitrary) numbers are relatively prime? He easily establishes that this probability is $\frac{1}{\zeta(2)}$ (where ζ is the Riemann zeta function), and after some pages of calculus shows that the numerical value of this expressions is $\frac{6}{\pi^2}$.

Chapters 11–12

Chapter 11 is about three upper bounds: if f is a differentiable function, R a parameter and $I_f(R) = \int_a^b \exp(iRf(x))dx$, then: if f' is strictly monotonic with $f'(x) \geq 1$, we have $|I_f(R)| \leq \frac{4}{R}$ (van der Corput theorem), and if $f''(x) \geq 1$, then the inequality is a bit weaker: $|I_f(R)| \leq \frac{10}{\sqrt{R}}$.

The third upper bound is about the characteristic function χ_D of the unit disk in the plane and its Fourier transform $\widehat{\chi}_D(\xi) = \int_D \exp(-2\pi i x\xi)dx$. The author shows that $|\widehat{\chi}_D(\xi)| \leq C|\xi|^{-\frac{3}{2}}$.

In Chapter 12, the author applies the results of Chapter 11 to the following setting: let $N(t) = \#\{tD \cap \mathbb{Z}^2\}$ be the integer points of the disk of radius t in $\mathbb{R}^2$. As we know from school, the area of tD is πt^2, and this number is an approximation of $N(t)$. It happens that many mathematicians have tried to find an upper bound of the difference $E(t) = N(t) - \pi t^2$.

Once again the author brilliantly shows how close a simple textbook exercise can be to a difficult result, and even to an unsolved century-old conjecture by Hardy (the British friend of legendary mathematician Ramanujan): the upper bound $|E(t)| \leq Ct$ is given as a simple exercise, $|E(t)| \leq Ct^{\frac{2}{3}}$ is proven (Sierpinski, 1903), and $\forall \varepsilon > 0\, \exists C_\varepsilon$ such that $|E(t)| \leq C_\varepsilon t^{\frac{1}{2}+\varepsilon}$ is Hardy's conjecture. It seems that the best known proven result (Heath-Brown) is $|E(t)| \leq Ct^{\frac{19}{15}}$ and the author "conjectures" that the Hardy "Holy Grail" conjecture will be proven sometime in the 24th century...

Chapter 13

In Chapter 13 the author returns to finite fields $\mathbb{F}_p$ and defines the NTT (number-theoretic Fourier transform) $\widehat{f}(m) = p^{-d} \sum_{x \in \mathbb{F}_p^d} \exp(-\frac{2\pi i}{p}(-x \cdot m))f(x)$. He proves the Fourier inversion formula, as well as the Plancherel theorem saying that the sum of squares of $\widehat{f}$ (taken over all elements of $\mathbb{F}_p^d$) is p^{-d} times the sum of squares of f.

The last move of the book is to establish a connection between the Fourier transform of a subset E of $\mathbb{F}_p^d$ and the cardinal of the set of points (x, y, x', y') of E^4 such that $x + y = x' + y'$:

$$\sum_{m\in\mathbb{F}_p^d} |\widehat{E}(m)|^4 = p^{-3d}\#\{(x, y, x', y') \in E^4 \mid x + y = x' + y'\}.$$

The author shows that the cardinal of this set takes $C(\#E)^2$ as upper bound. As an example of a set satisfying this inequality, one can take $E = \{x \in \mathbb{F}_p^d \mid x_d = x_1^2 + \cdots + x_{d-1}^2\}$, in this case $C = 1$.

These results lead to the fact that if A is a subset of $\mathbb{F}_p^d$ of reasonable dimension ($\sqrt{p} \leq \#A \leq Cp^{\frac{7}{10}}$) then

$$\max\{\#(A + A), \#(A \cdot A)\} \geq \frac{\sqrt{(\#A)^3}}{\sqrt[4]{p}}$$

which is given as an exercise, and proven in a 2007 paper [2] by the author *et al.*

3 Opinion

In one of his most famous poems, Cavafy says[13] "Ithaka gave you the beautiful journey. // Without her you would not have set out. // She has nothing left to give you now. // And if you find her poor, Ithaka won't have fooled you. // Wise as you will have become, so full of experience, // you will have understood by then what these Ithakas mean." This is also the method of this book: there is no unique great result, no spectacular finale—but there is a journey. A journey through several domains of mathematics, interrelated and interacting.

Clearly the book is intended for undergraduate students, and it could very well have been a transcription of lecture notes: several times the author uses a purely oral style, like on p. 36: "Suffering is unavoidable here... so please do not start complaining if you are not done in two or three hours..." Sometimes the author uses all-caps words and multiple exclamation marks for emphasis, like on p. 67: "DO NOT stop here! Always look for generalizations and variants! ALWAYS!! Yes, I am shouting... WORK IT OUT!!!" Due to the author's natural charisma, the book is pleasant to read. Furthermore, the fact of wandering through so many areas of mathematics makes one feel confident and opens new horizons.

These are the positive aspects of the method used. But one also could object that there are no solutions given for the many exercises, so one is entirely left on his/her own. Also it is not always clear for what reason the author examines specific topics. A typical example is the issue of calculating the dimension of unit ball projections needed to contain a unit square: what makes this an important problem in the first place?

As already mentioned, it is quite a thrill to discover that just by slightly changing a trivial result one gets hard problems and conjectures. Many statements in the book are Goldbach-like: trivially easy to formulate, and yet unsolvable, or, at least, unsolved up to date. But are they important because they carry the name of some famous mathematician?

Maybe the main interest of the book is to give a sense of unity of mathematics. Its title is "A View from the Top" and indeed one has the impression of being on the top of some skyscraper and looking at the various neighborhoods of a town (neighborhoods called calculus, probabilities, number theory, algebra, Fourier transform, etc.), watching people live their ways in every one of them, and discovering connections and similarities between them.

[13] Transl. Ed. Keeley & Ph. Sherrard, Princeton University Press, 2009.

I recommend this book to people who are curious about discovering things that are usually not taught, nontrivial interconnections and unexpected Goldbach-like conjectures. Time and energy will be needed to, at least, give a try to the many exercises. And the reader will have to try hard not to get frustrated with being unable to solve some/many of them.

The book finishes with a Knuthian exhortation which is so nice and so universal, that I can't avoid giving it here: "Can you anticipate further developments? Can you formulate key questions that could lead to further progress? Are you willing to tirelessly search the research journals and the internet to find out what the concepts you have been introduced are connected to? This book can only be called a success if it causes you to do all these things and more. Good luck!"

References

[1] R. Davies, "Some remarks on the Kakeya problem," *Proc. Camb. Phil. Soc.* **69** (1971), 417–421.

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[3] N. Katz and G. Tardos, "A new entropy inequality for the Erdős distance problem," *Contemp. Math.* **342** (2004).

[4] L. H. Loomis and H. Whitney, "An inequality related to the isoperimetric inequality," *Bull. Amer. Math. Soc.* **55** (1949), 961–962.

[5] E. Szemeredi and W. Trotter, "Extremal problems in discrete geometry," *Combinatorica* **3** (1983), 381–392.