Sampling on Manifolds

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Definition

Let $f : \mathbb{R}^d \to \mathbb{C}$. The **Fourier transform** *of f is defined by*

$$\hat{f}(m) = \int_{-\infty}^{\infty} f(x)\, \chi(-mx) dx, \quad \forall m \in \mathbb{R}^d.$$

where $\chi(t) = e^{(2\pi i t)}$.

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Remark (Fourier Inversion and Plancherel)

We have

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(m)\chi(xm)dm, \quad \text{and} \quad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\widehat{f}(m)|^2 dm.$$

Definition (Support)

Let $f : \mathbb{R}^d \to \mathbb{C}$, *then the support of* f *is* $\mathrm{supp}\,(f) = \{x \in \mathbb{R}^d : f(x) \neq 0\}$.

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Theorem (Classical Uncertainty Principle)

Let $f : \mathbb{R}^d \to \mathbb{C}$ *be supported in a subset* $E \subseteq \mathbb{R}^d$ *and let* $\widehat{f}$ *be supported in a subset* $S \subseteq \mathbb{R}^d$. *Then*

$$|E| \cdot |S| \geqslant c > 0,$$

where $|\cdot|$ *is the d-dimensional Lebesgue measure.*

Figure: Uncertainty Principle

source: https://brilliant.org/wiki/heisenberg-uncertainty-principle/

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Definition (Concentration)

Let (X, μ) be a measure space, and let $E \subset X$. We say that $f \in L^p(X, \mu)$ is *L^p-concentrated on E at level ϵ* if

$$\|f - \mathbf{1}_E f\|_{L^p(X)} \leqslant \epsilon \|f\|_{L^p(X)}, \quad \text{for some } 0 \leqslant \epsilon < 1. \tag{1}$$

Definition

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What is a Manifold?

Definition

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- Locally Euclidean, but the global shape can be curved or twisted.
- **Hausdorff Space:** the points can be "separated" in the topology
- **Charts:** coordinate maps from pieces of M to $\mathbb{R}^n$.
- **Second Countable:** there is a countable basis for the topology.
- **Locally Euclidean:** This means that $\forall\ p \in M$ there is some set open set $U \subset M$ where $p \in U$ and open $\hat{U} \subset \mathbb{R}^n$ such that $\mathbf{x} : U \to \hat{U}$ is a homeomorphism.
- Examples: $\mathbb{R}^n$, sphere S^n, torus $\mathbb{T}^n$.

DEFINITION

A **derivation**, v, at a point $p \in M$, is a linear function which assigns each smooth function on M to a real number in $\mathbb{R}$ in such a way that the product rule holds $(v(fg) = f(p)v(g) + g(p)v(f))$. The set of all such derivations is the **Tangent Space**, T_pM. This captures the idea of tangent vectors to a manifold.

In the case of surfaces from $\mathbb{R}^2$ to $\mathbb{R}^3$ dealt with in the examples later, there is an easier interpretation. Take some point $p \in M$, $w \in \mathbb{R}^2$, and some chart covering that point $\mathbf{x} : U \to \hat{U}$ where $\mathbf{x}(q) = p$. Then take some curve $\alpha : (-\varepsilon, \varepsilon) \to M$ such that $\alpha(0) = q$ and $\alpha'(0) = v$. Then, define $d\mathbf{x}_p(w) := (\mathbf{x} \circ \alpha)'(0)$. Then the tangent space is at a point p is given by $T_pM := d\mathbf{x}_p(\mathbb{R}^2)$.

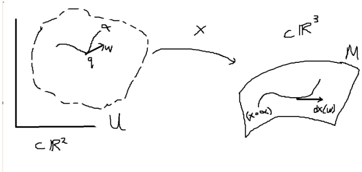

Figure: Showing how vectors are "pushed forward" from $\mathbb{R}^2$ to $\mathbb{R}^3$

DEFINITION

A **Riemannian manifold** (M, g) is a manifold equipped with a smoothly varying inner product g_p on each tangent space T_pM.

In further examples, the surfaces we deal with all have the property that, given the chart $\mathbf{x}(u, v)$ that $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent and form a basis for the tangent space. Since the inner product is a bilinear form, this has a symmetric matrix associated with it given by:

$$g_p = \begin{bmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_u, \mathbf{x}_v \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{bmatrix}$$

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- g lets us measure lengths, angles, and volumes.
- Examples:
  - S^2 with the round metric.
  - Flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with Euclidean metric.
- With g, we can define the Laplace–Beltrami operator.

DEFINITION

Let (M, g) be a compact Riemannian manifold without boundary. The **Laplace–Beltrami operator** is

$$\Delta_g f = |g|^{-1/2} \partial_i \left(|g|^{1/2} g^{ij} \partial_j f \right),$$

where (g^{ij}) is the inverse matrix and $|g|$ its determinant.

- Self-adjoint and negative-definite on $L^2(M)$.
- Eigenfunctions e_j satisfy $\Delta_g e_j = -\lambda_j^2 e_j$.

- For each eigenvalue λ, the **eigenspace**

$$E_\lambda = \{e_j : \lambda_j = \lambda\}$$

has dimension $\#\{j : \lambda_j = \lambda\}$ (the multiplicity of λ).

- $\{e_j\}$ can be chosen orthonormal in $L^2(M)$:

$$\int_M e_j(x)\, \overline{e_k(x)}\, dV_g(x) = \delta_{jk}.$$

- The pointwise sum

$$S_\lambda(x) = \sum_{j:\, \lambda_j = \lambda} |e_j(x)|^2$$

measures the total "energy density" of that eigenspace at x.

Theorem

Let S be a finite subset of the set of eigenvalues of $\sqrt{-\Delta_g}$. Let $X_S = \{j : \lambda_j \in S\}$. Suppose that $f \in L^2(M)$ is not identically zero and that f is L^2-concentrated in $E \subset M$ at level L with respect to the Riemannian volume density. Suppose also that $\hat{f}$ is L^2-concentrated on X_S at level L' with respect to the counting measure. Then

$$\left(\frac{1}{\# X_S} \sum_{j \in X_S} \frac{1}{|E|} \int_E |e_j(x)|^2 \, dx \right)^{-1} \leqslant \frac{|E| \cdot \# X_S}{(1 - \epsilon - \epsilon')^2},$$

where

$$L = (1 - \epsilon^2)^{-1/2} \quad \text{and} \quad L' = (1 - \epsilon'^2)^{-1/2}.$$

Main equation

For each eigenvalue λ,

$$\sum_{j\colon \lambda_j = \lambda} |e_j(x)|^2 \equiv c \quad \text{a.e. } x \in M,$$

with c independent of x.

This, thus will lead to the following relation:

Relation

$$\frac{1}{\#\{j : \lambda_j \in S\}} \sum_{\lambda_j \in S} \frac{1}{|E|} \int_E |e_j(x)|^2 = \frac{1}{|M|}$$

Combined with the aforementioned theorem, we get an alternative version of the uncertainty principle below:

Altered uncertainty principle

$$(1 - \epsilon - \epsilon')^2 \leqslant \frac{|E|}{|M|} \cdot \#X_S$$

- Our goal is to understand when the constant-sum property

$$S_\lambda(x) = \sum_{j:\, \lambda_j = \lambda} |e_j(x)|^2 \quad \text{is constant in } x$$

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  holds.
- Rather than start with a general manifold, we first explore *simple, symmetric surfaces*:
  - **Flat torus** $\mathbb{T}^2$.
  - **Sphere** S^2.
  - **Spheroids**.

1. Choose a surface.
2. Compute L^2-normalized eigenfunctions for a fixed eigenvalue.
3. Evaluate $\sum_{j:\, \lambda_j = \lambda} |e_j(x)|^2$ on a grid for some fixed λ.
4. Test if the range is constant.

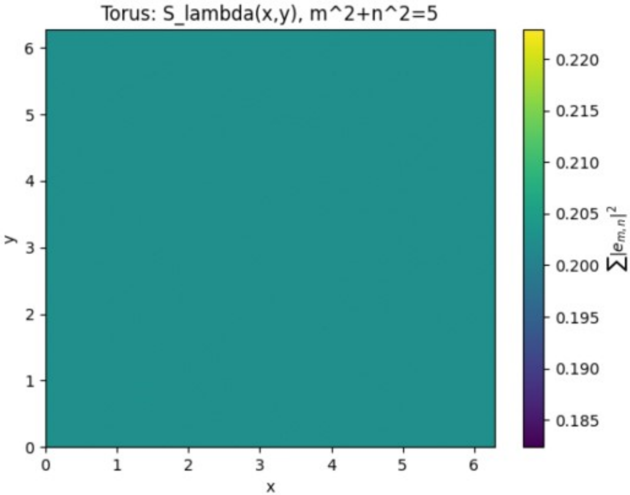

Figure: Sum over a torus

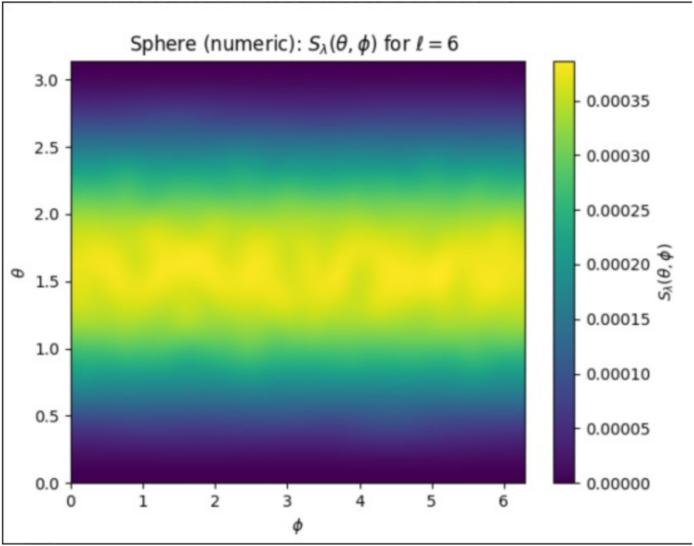

FIGURE: Sum over a sphere

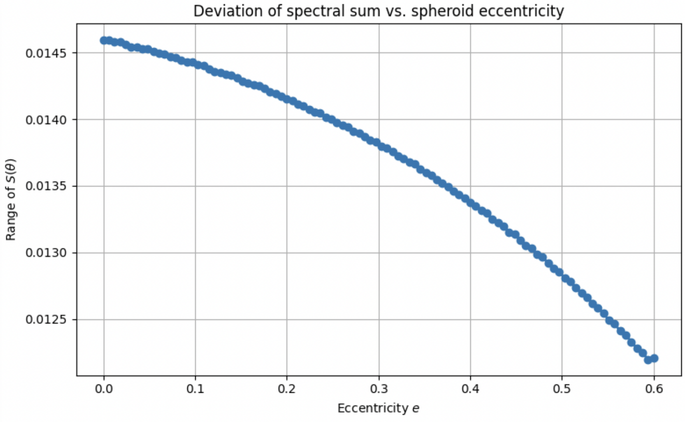

Figure: Sum over a spheroid

We conjecture that

$$\sum_{j:\,\lambda_j=\lambda} |e_j(x)|^2 \equiv c \quad \text{a.e. } x \in M,$$

For this family of surfaces as we see that we stay somewhat constant throughout these three surfaces, but more data testing for other surfaces is needed before proving this conjecture.