Cycle Covers 2025

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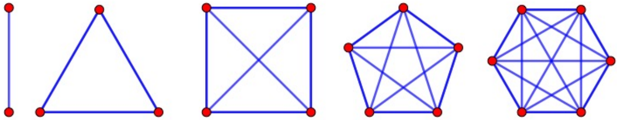
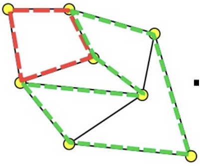
What is a graph?

- A **graph** G can be represented as a set of n vertices and m edges. If a pair of vertices, say u and v, are adjacent, we can represent the edge between them as $uv = vu$.
- $G = (V, E)$
- $V(G) = \{v_i | i \in [n]\}$
- $E(G) = \{v_i v_j : v_i \text{ is adjacent to } v_j\}$

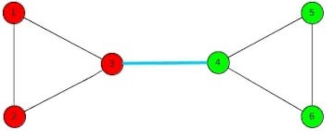


Complete Graphs

- A **cycle** C can be defined as a sequence of adjacent vertices such that that first and last vertex are the same and no other vertices are repeated.



- A **bridge** is an edge such that if deleted, would separate the graph into two distinct sections.



- It is easy to see that a bridge can not be part of a cycle.

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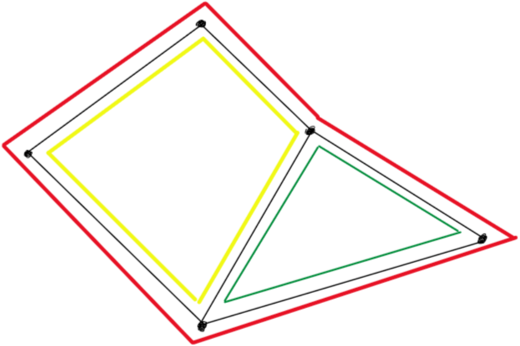
Cycle Double-Cover Conjectures

- **Cycle Double-Cover Conjecture (CDCC):** If G is a bridgeless graph, then there exists a collection of cycles such that every edge is used exactly twice.

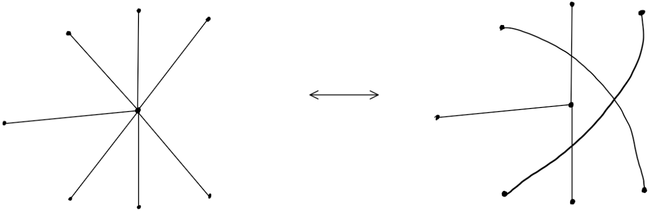
- **Strong Cycle Double-Cover Conjecture (SCDCC):** If G is a bridgeless graph and C a cycle in G, then there exists a collection of cycles containing C such that every edge is used exactly twice.

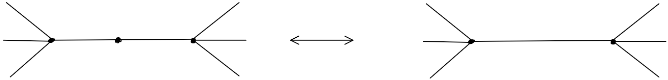
Cycle covers from planar graphs

- Given a bridgeless planar graph, a CDC can be constructed by taking the boundary of each face as a cycle.



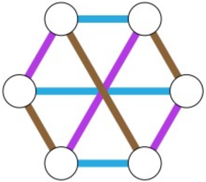
Reduction to Cubic Graphs





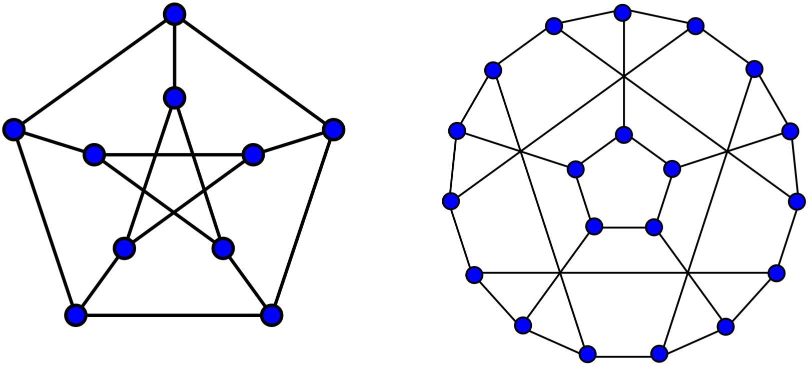
Cycle covers from 3-edge coloring of cubic graph

- Given a cubic graph, if there exists a 3-edge coloring, we can a CDC of the graph by taking each possible pair of colors to be a collection of cycles.



3-edge coloring

- Thus, the CDCC can be reduced to cubic graphs whose edges can not be 3-edge colored. These graphs are known as **snarks**.



Examples of snarks

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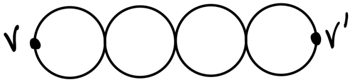
Minimal Cycle Path

- **Cycle Path:** A collection of cycles connecting two given vertices
- **Minimal Cycle Path:** A minimal cycle path is cycle path we can represent as a sequence $C_1, ...C_m$ of cycles such that cycles only share edges/vertices with the cycles adjacent in the sequence.

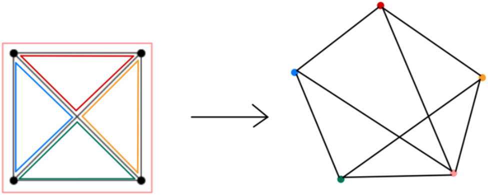


Minimal Cycle Path from v to v'

Cycle Cover Dual

We consider minimal cycle paths formed from the cycles in an existing double cover, we can first construct the following:

- Given a (bridgeless, subcubic) graph G and a cycle double cover S, we can form the cycle cover dual graph, called $D_S(G)$.
- This graph is made by creating a node for every cycle in S and connecting two nodes if those two corresponding cycles share an edge in G. (i.e. it is the intersection graph of the edge sets of cycles in S)

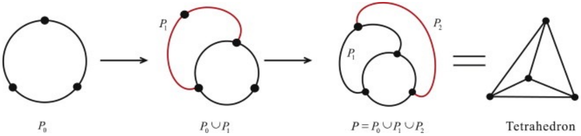


A graph G with CDC S and $D_S(G)$

- To find an MCP in G using cycles from S we simply need to find an induced path in $D_S(G)$.

Ear decomposition

- Given a 2-connected graph G, we can start with a cycle and attach paths by identifying their end vertices with distinct vertices in the graph. This is called adding an **ear**.



- Given an applicable graph with an existing CDC and an added ear, if we can solve a soon to be mentioned subproblem, then we can adjust the CDC to accommodate the added ear.
- Using this algorithm, one can inductively construct CDCs (and SCDCs) if one can solve the aforementioned subproblem.

Why MCPs are useful for Cycle Double Cover

The CDCC requires that each edge is in exactly two cycles:

- Say we have a bridgeless (sub)cubic graph G with a CDC S.
- Then suppose we add an edge between two degree 2 vertices v, v'.
- Can we modify the existing CDC to get a cover for this new graph?
- Intuitively, in general we should be able to leave most of the cycles alone and modify a small number of them to accommodate this new edge.
- We can find a v, v' MCP using cycles $M \subset S$.
- Let $G[M]$ be the subgraph with edges $\bigcup_{c \in M} E(c)$. If we add the edge vv' to this and can find a $\{1, 2\}$-cycle cover K of it using vv' twice and all other edges the number of times they were used by cycles in M, then $(S \backslash M) \cup K$ is a CDC of $G + vv'$.
- Does every CDC of such a graph have an MCP that is "solvable" in the above sense for any degree 2 v, v'?

Algorithm 1: Get Cycle Cover Dual

Input: A graph G, a list of cycles $\mathcal{L} = [L_1, L_2, \ldots, L_n]$
Output: A cycle cover dual H where each node represents a cycle in $\mathcal{L}$

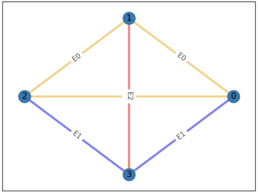
```
Initialize an empty graph H ;
for i ← 1 to n − 1 do
    for j ← i + 1 to n do
        if L_i ∩ L_j ≠ ∅ then
            Add edge (i, j) to H ;      // Add edge if cycles intersect
return H
```

Algorithm 2: Get Minimal Cycle Path

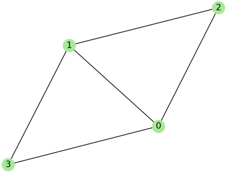
Input: A graph G, a list of cycles $\mathcal{L} = [L_1, L_2, \ldots, L_n]$, and two degree-2 vertices v_1, v_2

Output: A minimal cycle path between v_1 and v_2

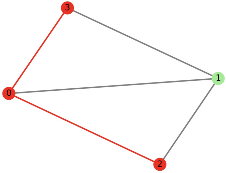
1. Use Algorithm 1 to find the cycle cover dual H;
2. $C_1 =$ a cycle that contains v_1 ;
3. $C_2 =$ a cycle that contains v_2 ; // $C_1, C_2 \in \mathcal{L}$
4. MCP $=$ induced path between C_1 and C_2 in cycle cover dual H (up to possibly removing first or last edge)



Ear decomposition of K_4



Intersection graph of a double cover of K_4



Induced path in the intersection graph of a double cover of K_4

- If we are at step i in the ear decomposition, we have a graph G_i and a CDC S_i. Let n_i be the number of vertices at step i and m_i be the number of edges at step i.

Form intersection graph of $E(c_1), E(c_2), \ldots$	$\mathcal{O}(m_i^2)$
Find induced paths between two vertices	$\mathcal{O}(n_i + m_i)$
Take corresponding cycles in $M_i \subseteq S_i$ and form $G[M]$	$\mathcal{O}(m_i)$
*Get 2-edge coloring of $r(G[M_i]) - X_{M_i}$	$\mathcal{O}(n_i + m_i)$
*Insert v_1, v_2 and $v_1 v_2$ and adjust coloring	$\mathcal{O}(n_i + m_i)$
*Form cycles of K_i from coloring	$\mathcal{O}(1)$
Modify existing cover to $(S_i / M_i) \cup K_i$	$\mathcal{O}(1)$

- $*=$ steps only for the implemented (easiest) case

- The time complexity at step i is bounded by $\mathcal{O}(n_i + m_i^2)$. Let $j = m - n + 1$. Then across every step from $i = 1$ to $i = j = (m - n + 1)$, the time complexity is then bounded by:

$$j \cdot \mathcal{O}(n_j + m_j^2) \leq \mathcal{O}(n_j m_j + m_j^3 - n_j^2 - n_j m_j^2) \leq \mathcal{O}(m_j^3)$$

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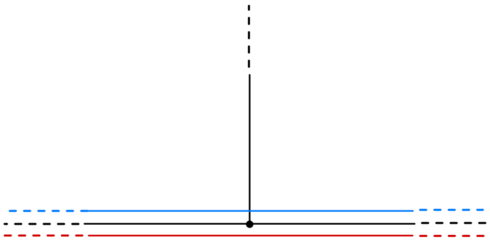
Compatible Cycles

- **Def:** If C and C' are two cycles in a bridgeless graph, we say they're **compatible** if they share edges but not consecutive edges.



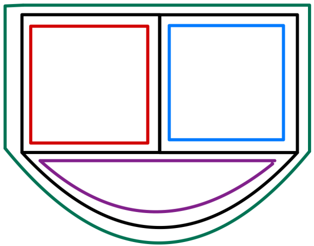
Non-compatible cycles

- With bridgeless cubic G, if two cycles share edges but aren't compatible, then they can't be in a CDC of G together.

Highly Compatible Cycles

- **Def:** If two cycles C, C' are compatible and $C \Delta C'$ is connected (i.e., a single cycle as opposed to a disjoint union of cycles), then we say C and C' are **highly compatible**.

- Note that $C \Delta C'$ refers to the subgraph with edge set $E(C) \Delta E(C')$



Graph with a CDC

- **Conjecture:** In a bridgeless cubic graph G every cycle C has at least one **compatible** cycle C'.
- **Conjecture:** In a bridgeless cubic graph G every cycle C in has at least one **highly compatible** cycle C'.

- First, we have that

  Conj 2 $\Rightarrow$ **SCDCC** $\Rightarrow$ **Conj 1**

  we also show the following:

- **Theorem:** If the SCDCC is true, then given two highly compatible cycles, we can find a double cover containing both.

Minimal Counterexample to Conj 2

- **Theorem:** If (G, C) is a minimal counterexample to **Conj 2**, then C is a dominating cycle, and the SCDC conjecture holds for all graphs smaller than G.
- C is **dominating** if every vertex in G is either in C or adjacent to a vertex in C.

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Future Goals

- There are other topics, such as semiextensions and splits of cycles that have strong ties to compatible cycles. We would like to further explore their connections.
- Ideally we would like to solve the core subproblem and turn this into a complete solution.