Forecastability of Time-Series and Additive Combinatorics

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Definition

Let $f : \mathbb{Z}_N \to \mathbb{C}$. *The* **Fourier transform** *of* f *is defined by*

$$\widehat{f}(m) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}_N} f(x)\chi(-xm)$$

where $\chi(t) = \exp\left(\frac{2\pi i t}{N}\right)$.

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Remark (Fourier Inversion and Plancherel)

We have

$$f(x) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}_N} \widehat{f}(m)\chi(xm), \quad \textit{and} \quad \sum_{x \in \mathbb{Z}_N} |f(x)|^2 = \sum_{m \in \mathbb{Z}_N} |\widehat{f}(x)|^2.$$

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Definition

Given $g : \mathbb{Z}_N \to \mathbb{C}$, we define the L^p norms

$$\|g\|_{L^p(\mu)} = \left(\frac{1}{N} \sum_{x \in \mathbb{Z}_N} |g(x)|^p\right)^{\frac{1}{p}}, \quad \|g\|_p = \left(\sum_{x \in \mathbb{Z}_N} |g(x)|^p\right)^{\frac{1}{p}}, \quad \textit{and } \|g\|_\infty = \max_{x \in \mathbb{Z}_N} |g(x)|.$$

- We want to understand random, or "unstructured" sets, so we make the following definition:

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Definition

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- The following result due to Bourgain and Talagrand describes the behavior of generic sets.

Theorem (Bourgain and Talagrand, 1998)

There exists $\gamma_0 \in (0,1)$ such that if $h : \mathbb{Z}_N \to \mathbb{C}$ supported in a generic set M of size $\gamma_0 \frac{N}{\log(N)}$, then with probability $1 - o_N(1)$,

$$\left(\frac{1}{N} \sum_{m \in \mathbb{Z}_N} |\widehat{h}(m)|^2 \right)^{\frac{1}{2}} \leqslant C_T (\log(N) \log\log(N))^{\frac{1}{2}} \cdot \frac{1}{N} \sum_{m \in \mathbb{Z}_N} |\widehat{h}(m)|, \tag{1}$$

where $C_T > 0$ is a constant that depends only on γ_0.

- Similarly, we have the following theorem due to Bourgain:

THEOREM (BOURGAIN, 1989)

Suppose that M is generic with $|M| = \lceil N^{\frac{2}{q}} \rceil$, $q > 2$. Then with high probability, for all $h : \mathbb{Z}_N \to \mathbb{C}$ supported in M,

$$\left(\frac{1}{N} \sum_{m \in \mathbb{Z}_N} \left| \widehat{h}(m) \right|^q \right)^{\frac{1}{q}} \leqslant C(q) \cdot \left(\frac{1}{N} \sum_{m \in \mathbb{Z}_N} \left| \widehat{h}(m) \right|^2 \right)^{\frac{1}{2}}$$

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- An application of Hölder's inequality to the above theorem shows that if $|M| = O(N^{1-\epsilon})$, then Talagrand's inequality holds without the logarithms:

COROLLARY

If M is generic with $|M| = \lceil N^{\frac{2}{q}} \rceil$, $q > 2$, then with high probability, for all $h : \mathbb{Z}_N \to \mathbb{C}$ supported in M,

$$\left(\frac{1}{N} \sum_{x \in \mathbb{Z}_N} \left| \widehat{h}(x) \right|^2 \right)^{\frac{1}{2}} \leqslant (C(q))^{\frac{q}{q-2}} \cdot \frac{1}{N} \sum_{x \in \mathbb{Z}_N} |\widehat{h}(x)|.$$

- First, we experimentally estimate the constants C_T and $C(q)$.

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- For generic M of size $O(N^{\frac{2}{q}})$ $(q \geqslant 2)$, Talagrand's theorem states that for h supported in M,

$$\|\widehat{h}\|_{L^2(\mu)} \leqslant C_T \|\widehat{h}\|_{L^1(\mu)},$$

  while Bourgain's theorem states

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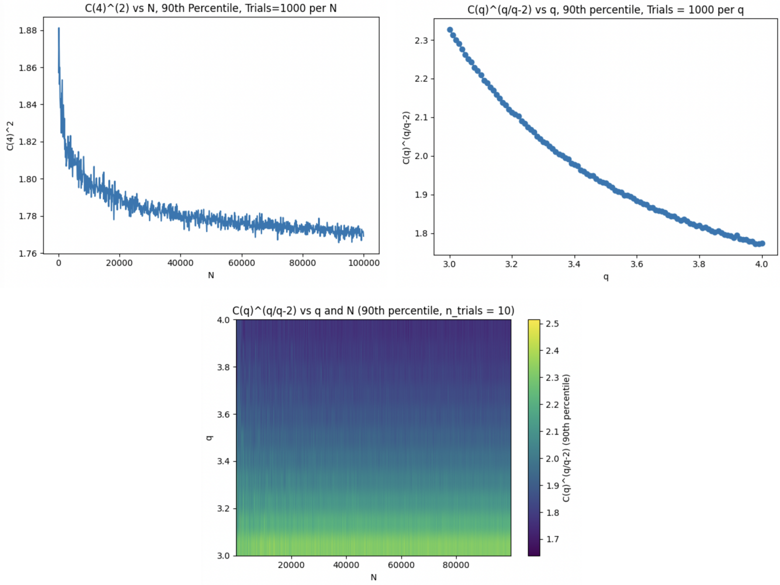
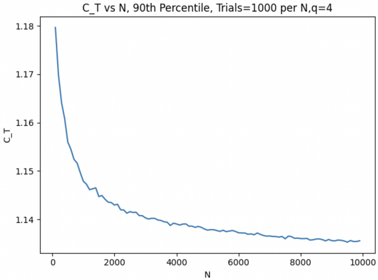
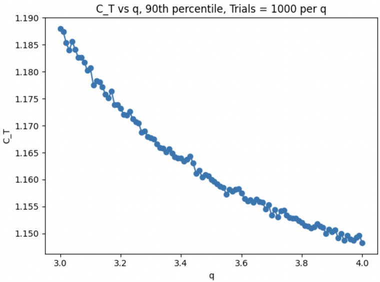
while Bourgain's theorem states
$$\|\widehat{h}\|_{L^q(\mu)} \leqslant C(q) \|\widehat{h}\|_{L^2(\mu)}.$$
- We saw also that $C_T \leqslant C(q)^{\frac{q}{q-2}}$, so we estimate and compare these two quantities.

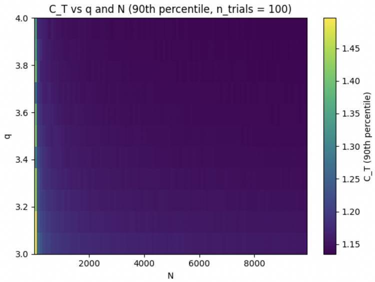
- Looking at the values for $C(q)^{\frac{q}{q-2}}$, we see it is roughly between 1.5 and 2.5:



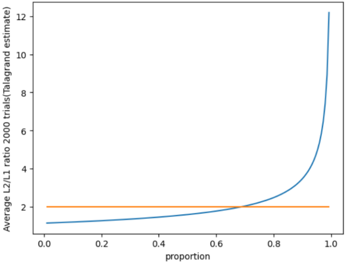
- Next, looking at the values for C_T, we see it is smaller, roughly between 1 and 1.2:

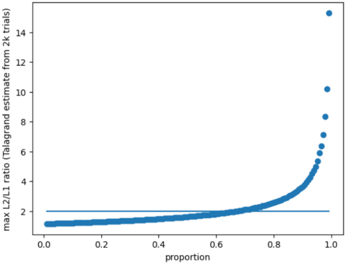


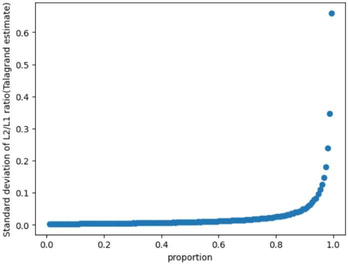




Talagrand's Constant







- From Bourgain and Talagrand, if the support of h is generic, then we have

$$\frac{\|\widehat{h}\|_{L^1(\mu)}}{\|\widehat{h}\|_{L^2(\mu)}} \geqslant \frac{1}{C_T}.$$

This implies the ratio $\frac{\|\widehat{h}\|_{L^1(\mu)}}{\|\widehat{h}\|_{L^2(\mu)}}$ is large.

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- Since C_T is close to 1, this suggests the ratio $\frac{\|\widehat{h}\|_{L^1(\mu)}}{\|\widehat{h}\|_{L^2(\mu)}}$ is close to 1 when h has random support.

- On the other hand, when $N = pq$ and S is a subgroup of order p, we can show that for $h = 1_S$,

$$\frac{\|\widehat{h}\|_{L^1(\mu)}}{\|\widehat{h}\|_{L^2(\mu)}} = \frac{1}{\sqrt{p}}.$$

- On the other hand, when $N = pq$ and S is a subgroup of order p, we can show that for $h = 1_S$,

$$\frac{\|\widehat{h}\|_{L^1(\mu)}}{\|\widehat{h}\|_{L^2(\mu)}} = \frac{1}{\sqrt{p}}.$$

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- When p is large, this gives a small ratio, while when p is small, the ratio is close to 1.
- For p large, this suggests dense structured sets have small ratio, while sparse sets have ratio close to 1.

- By Cauchy-Schwartz,

$$\frac{1}{N}\sum_{m\in\mathbb{Z}_N}|\widehat{f}(m)| \leqslant \left(\frac{1}{N}\sum_{m\in\mathbb{Z}_N}|\widehat{f}(m)|^2\right)^{\frac{1}{2}},$$

and thus $\frac{\|\widehat{f}\|_{L^1(\mu)}}{\|\widehat{f}\|_{L^2(\mu)}} \leqslant 1$.

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and thus $\frac{\|\widehat{f}\|_{L^1(\mu)}}{\|\widehat{f}\|_{L^2(\mu)}} \leqslant 1$.

- Moreover, by Fourier inversion and the triangle inequality,

$$|f(x)| \leqslant \frac{1}{\sqrt{N}}\sum_{m\in\mathbb{Z}_N}|\widehat{f}(m)|,$$

and thus $\|f\|_2 \leqslant \|\widehat{f}\|_1$, so that $\frac{\|\widehat{f}\|_{L^1(\mu)}}{\|\widehat{f}\|_{L^2(\mu)}} \geqslant \frac{1}{\sqrt{N}}$.

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and thus $\|f\|_2 \leqslant \|\widehat{f}\|_1$, so that $\frac{\|\widehat{f}\|_{L^1(\mu)}}{\|\widehat{f}\|_{L^2(\mu)}} \geqslant \frac{1}{\sqrt{N}}$.

Definition (The Fourier Ratio)

We thus define

$$FR(f) = \sqrt{N}\cdot\frac{\|\widehat{f}\|_{L^1(\mu)}}{\|\widehat{f}\|_{L^2(\mu)}}.$$

- We show that if $FR(f)$ is small, f is structured.

- We compute $FR(f)$ for the following datasets:

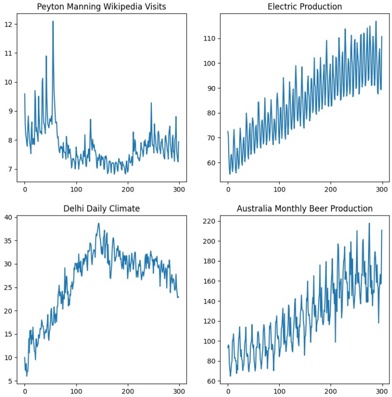

Figure: The four data sets.

Computing $FR(f)$ for structured f

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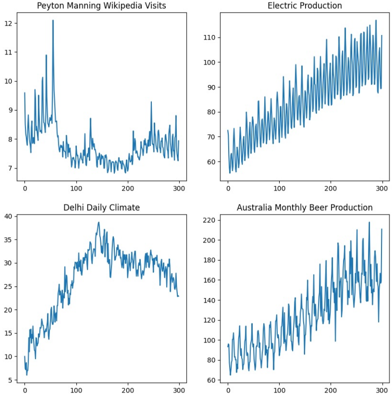


FIGURE: The four data sets.

The results are as follows:

- Peyton Manning Wikipedia Visits:

$$FR(f) = 1.917$$

- Electric Production:

$$FR(f) = 2.133$$

- Delhi Daily Climate:

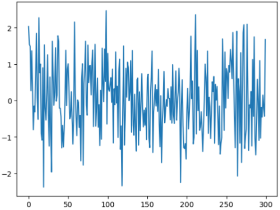
$$FR(f) = 2.715$$

- Australia Monthly Beer Production:

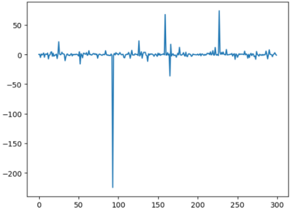
$$FR(f) = 2.884$$

Note these are all close to the minimal value $FR(f) = 1$.

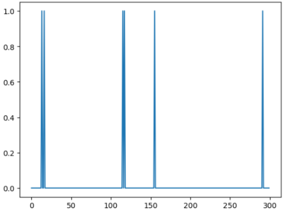
- Next we compute $FR(f)$ when $f(x)$ is randomly chosen according to various distributions:



Ratio for normal distribution: 15.13



Ratio for cauchy distribution: 16.37



Ratio for poisson distribution: 15.52

- Note that in all cases, $FR(f)$ is close to $\sqrt{N} \approx 17.32$

- Our first result shows that concentration in a random set is good enough to have large $FR(f)$.

THEOREM

Let $f : \mathbb{Z}_N \to \mathbb{C}$. Suppose that there exists a generic set M such that

$$\|f\|_{L^2(M^c)} \leqslant r\|f\|_2$$

for some $r \in (0,1)$, with $|M| \leqslant \gamma_0 \frac{N}{\log(N)}$, where γ_0 is as in Talagrand's theorem. Suppose that

$$r < \frac{1-r}{C_T\sqrt{\log(N)\log\log(N)}}. \tag{2}$$

Then

$$\frac{\|\widehat{f}\|_{L^1(\mu)}}{\|\widehat{f}\|_{L^2(\mu)}} \geqslant \frac{1-r}{C_T\sqrt{\log(N)\log\log(N)}} - r \tag{3}$$

with probability $1 - o_N(1)$.

- Next, we show that f can be approximated by a polynomial with degree depending on $FR(f)$.

Theorem

Let $f : \mathbb{Z}_N \to \mathbb{C}$ and $\eta > 0$. Then for any

$$k > \frac{FR(f)^2 - 1}{\eta^2},$$

there exists a trigonometric polynomial

$$P(x) = \sum_{i=1}^{k} c_i \chi(m_i x)$$

such that

$$\|f - P\|_2 < \eta \|f\|_2.$$

- The proof is probabilistic, and shows that when $FR(f)$ is small, f can be approximated by a low degree polynomial.

- Similar methods also give an L^∞ approximation, now in terms of the ratio $\frac{\|\widehat{f}\|_{L^1(\mu)}}{\|f\|_\infty}$

Theorem

Let $f : \mathbb{Z}_N \to \mathbb{C}$ and let $\eta > 0$. Then for any k such that

$$k > 8\left(\frac{\|\widehat{f}\|_{L^1(\mu)}}{\|f\|_\infty}\right)^2 \frac{N\log(4N)}{\eta^2},$$

there exists a trigonometric polynomial

$$P(x) = \sum_{i=1}^{k} c_i \chi(m_i x)$$

such that

$$\|f - P\|_\infty < \eta\|f\|_\infty.$$

- Note that the triangle inequality shows $\frac{\|\widehat{f}\|_{L^1(\mu)}}{\|f\|_\infty} \geqslant N^{-\frac{1}{2}}$, and so in the best case, the above theorem gives a polynomial of degree $O(\log(N))$.

- Suppose $f : \mathbb{Z}_N \to \mathbb{C}$.

DEFINITION (RANDOM FUNCTION)

Define the random function $Z : \mathbb{Z}_N \to \mathbb{C}$, where for each m,

$$Z(x) = \|\widehat{f}\|_1 \operatorname{sgn}(\widehat{f}(m)) N^{-\frac{1}{2}} \chi(mx)$$

with probability $\frac{|\widehat{f}(m)|}{\|\widehat{f}\|_1}$, where $\operatorname{sgn}(z) = \frac{z}{|z|}$.

Proof of L^2 Approximation

- Suppose $f : \mathbb{Z}_N \to \mathbb{C}$.

Definition (Random Function)

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- Note: We have that $\mathbb{E}[Z(x)] = f(x)$.
- And,

$$\begin{aligned}
\mathbb{E}|Z(x)|^2 &= \sum_{m \in \mathbb{Z}_N} \left| \|\widehat{f}\|_1 \frac{\widehat{f}(m)}{|\widehat{f}(m)|} N^{-\frac{1}{2}} \chi(mx) \right|^2 \cdot \frac{|\widehat{f}(m)|}{\|\widehat{f}\|_1} \\
&= \frac{1}{N} \|\widehat{f}\|_1 \sum_{m \in \mathbb{Z}_N} |\widehat{f}(m)| \\
&= \frac{1}{N} \|\widehat{f}\|_1^2,
\end{aligned}$$

- The Variance of $Z(x)$ can be shown to be,

$$\mathrm{Var}(Z(x)) = \mathbb{E}|Z(x)|^2 - |\mathbb{E}[Z(x)]|^2 = \frac{1}{N}\|\widehat{f}\|_1^2 - |f(x)|^2.$$

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- Let $Z_1, \ldots, Z_k$ be random i.i.d. functions with distribution Z, and define the random trigonometric polynomial P by

$$P(x) = \frac{1}{k}\sum_{i=1}^{k} Z_i(x).$$

- By linearity of expectation,

$$\mathbb{E}[P(x)] = \mathbb{E}[Z(x)] = f(x)$$

- Therefore, by independence

$$
\begin{aligned}
\mathbb{E}\|f - P\|_2^2 &= \sum_{x \in \mathbb{Z}_N} \mathbb{E}|f(x) - P(x)|^2 \\
&= \sum_{x \in \mathbb{Z}_N} \mathrm{Var}(P(x)) \\
&= \frac{1}{k} \sum_{x \in \mathbb{Z}_N} \mathrm{Var}(Z(x)) \\
&= \frac{1}{k} \sum_{x \in \mathbb{Z}_N} \frac{1}{N} \|\widehat{f}\|_1^2 - |f(x)|^2 \\
&= \frac{1}{k} \left(\|\widehat{f}\|_1^2 - \|f\|_2^2 \right).
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&= \frac{1}{k}\left(\|\widehat{f}\|_1^2 - \|f\|_2^2\right).
\end{aligned}
$$

- We finish with an assumption on the k value.

Proof of L^2 Approximation

- Now, if we assume this final value is less than $\eta^2\|f\|_2^2$, then there exists a deterministic choice of P such that $\|f - P\|_2 < \eta\|f\|_2$.

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- This assumption on k amounts to

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\begin{aligned}
k &> \frac{1}{\eta^2}\cdot\frac{\|\widehat{f}\|_1^2-\|f\|_2^2}{\|f\|_2^2} \\
&= \frac{1}{\eta^2}\left(\frac{\|\widehat{f}\|_1^2}{\|f\|_2^2}-1\right) \\
&= \frac{1}{\eta^2}\left(N\left(\frac{\|\widehat{f}\|_{L^1(\mu)}}{\|\widehat{f}\|_{L^2(\mu)}}\right)^2-1\right) \\
&= \frac{FR(f)^2-1}{\eta^2},
\end{aligned}
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- Now, if we assume this final value is less than $\eta^2\|f\|_2^2$, then there exists a deterministic choice of P such that $\|f - P\|_2 < \eta\|f\|_2$.
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&= \frac{1}{\eta^2}\left(N\left(\frac{\|\widehat{f}\|_{L^1(\mu)}}{\|\widehat{f}\|_{L^2(\mu)}}\right)^2 - 1\right) \\
&= \frac{FR(f)^2 - 1}{\eta^2},
\end{aligned}
$$

- and thus for any such k, there is a trigonometric polynomial P with $\|f - P\|_2 < \eta\|f\|_2$, and we are done. □

Theorem (Sparse-spectrum approximation from small $FR(f)$)

Let $f : \mathbb{Z}_N \to \mathbb{C}$, and fix $\eta > 0$. Define the large spectrum

$$\Gamma := \left\{ m \in \mathbb{Z}_N \ : \ |\widehat{f}(m)| \geqslant \eta \|f\|_{L^2(\mu)} \right\}.$$

Then

$$|\Gamma| \leqslant \frac{FR(f)}{\eta}\sqrt{N},$$

and moreover if

$$P(x) := \frac{1}{N^{\frac{1}{2}}} \sum_{m \in \Gamma} \widehat{f}(m)\chi(xm),$$

then

$$\|f - P\|_2 \leqslant \eta \|f\|_2.$$

In other words, f can be approximated by a polynomial of degree

$$\frac{FR(f)}{\eta} \cdot \sqrt{N}$$

up to an error $\leqslant \eta$.

- Let Γ be defined as above. By our assumption on f as well as Markov's inequality we have that

$$FR(f)\|f\|_2 \geqslant \sum_{m\in\mathbb{Z}_N} |\widehat{f}(m)| \geqslant \eta\|f\|_{L^2(\mu)}|\Gamma|,$$

- and thus if f is nonzero,

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- and thus if f is nonzero,
$$|\Gamma| \leqslant \frac{FR(f)}{\eta}\sqrt{N}.$$
- Next, defining P as above, note that P is the inverse Fourier transform of $\widehat{f}\cdot 1_\Gamma$. Additionally, for $m\notin\Gamma$, we have that
$$|\widehat{f}(m)| < \eta\frac{\|f\|_2}{N^{\frac{1}{2}}}.$$

- Let Γ be defined as above. By our assumption on f as well as Markov's inequality we have that

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$$|\widehat{f}(m)| < \eta\frac{\|f\|_2}{N^{\frac{1}{2}}}.$$

- Thus, by Plancherel we have that

$$\begin{aligned}\|f - P\|_2 &= \|\widehat{f} - \widehat{P}\|_2 \\ &= \|\widehat{f} - \widehat{f}1_\Gamma\|_2 \\ &= \left(\sum_{m\notin\Gamma} |\widehat{f}(m)|^2\right)^{\frac{1}{2}} \\ &\leqslant \eta\|f\|_2.\end{aligned}$$

Lemma (Chang's lemma)

Let $A \subset \mathbb{Z}_N$ have density $\alpha = \frac{|A|}{N}$, and for $\eta > 0$ define the large spectrum set

$$\Gamma = \left\{ m \in \mathbb{Z}_N \; : \; |\widehat{1_A}(m)| \geqslant \eta \alpha N^{\frac{1}{2}} \right\}.$$

Then there exists $\Lambda \subset \Gamma$ with

$$|\Lambda| \leqslant C\eta^{-2} \log\left(\frac{1}{\alpha}\right)$$

such that every $m \in \Gamma$ is a $\{-1, 0, 1\}$-linear combination of elements of Λ.

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such that every $m \in \Gamma$ is a $\{-1, 0, 1\}$-linear combination of elements of Λ.

- This indicates that when α is small, the set Γ should have some additive structure, since it is spanned by a small set.

Theorem (Generalized Chang's Lemma)

Let $f : \mathbb{Z}_N \to \mathbb{C}$*, and for* $\eta > 0$ *define the large spectrum set*

$$\Gamma = \left\{ m \in \mathbb{Z}_N \ : \ |\hat{f}(m)| \geqslant \eta \|f\|_{L^2(\mu)} \right\}.$$

Then there exists $\Lambda \subset \Gamma$ *with*

$$|\Lambda| \leqslant C\eta^{-2} \left(\frac{\|f\|_{\frac{\log N}{\log N - 1}}}{\|f\|_2} \right)^2 \log N \tag{4}$$

and

$$|\Lambda| \leqslant C\eta^{-2} \frac{\|f\|_1}{\|f\|_2} \log \left(\left(\frac{\|f\|_2}{\|f\|_1} \right)^2 N \right) \tag{5}$$

such that every $m \in \Gamma$ *is a* $\{-1, 0, 1\}$*-linear combination of elements of* Λ*.*

- Applying the previous theorem to $\hat{f}$, we obtain

$$|\Lambda| \leqslant C\eta^{-2} FR(f) \log\left(FR(f)^{-2} N\right),$$

where Λ is a set such that every

$$x \in \Gamma \coloneqq \left\{x \in \mathbb{Z}_N : |f(x)| \geqslant \eta \|f\|_{L^2(\mu)}\right\}$$

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- This suggests additive structure in the set Γ when $FR(f)$ is small.

- Let

$$g(x) = \frac{1}{\|1_\Lambda \hat{f}\|_2} \sum_{n\in\Lambda} \hat{f}(n)\chi(xn),$$

- Take $p > 2$, and let $p' = \frac{p}{p-1}$. Then

$$\begin{aligned}
\|f\|_{L^{p'}(\mu)}\|g\|_{L^p(\mu)} &\geqslant \|fg\|_{L^1(\mu)} \\
&\geqslant \frac{1}{N\|1_\Lambda \hat{f}\|_2}\left|\sum_{x\in\mathbb{Z}_N} f(x) \sum_{n\in\Lambda} \overline{\hat{f}(n)}\chi(-xn)\right| \\
&= \frac{1}{N\|1_\Lambda \hat{f}\|_2}\left|\sum_{n\in\Lambda} \overline{\hat{f}(n)} \sum_{x\in\mathbb{Z}_N} f(x)\chi(-xn)\right| \\
&= \frac{\sqrt{N}}{N\|1_\Lambda \hat{f}\|_2}\sum_{n\in\Lambda} |\hat{f}(n)|^2 \\
&= \frac{1}{\sqrt{N}}\|1_\Lambda \hat{f}\|_2.
\end{aligned}$$

- Recall that $|\hat{f}(m)| \leqslant \eta \frac{\|f\|_2}{N}$ for $m \in \Lambda$. Since we are summing over $|\Lambda|$ such terms,

$$\frac{1}{\sqrt{N}} \|1_\Lambda \hat{f}\|_2 \geqslant \eta \|f\|_2 \frac{\sqrt{|\Lambda|}}{N}$$

- This gives a lower bound on $\|f\|_{L^{p'}(\mu)} \|g\|_{L^p(\mu)}$.
- Next, we will establish an upper bound.

- By definition,

$$\|f\|_{L^{p'}(\mu)} = N^{\frac{1}{p}-1}\|f\|_{p'}.$$

- By Rudin's inequality (see Lemma 4.33 in Tao-Vu), we also have

$$\|g\|_{L^p(\mu)} \leqslant C\sqrt{p}$$

- Combining the our bounds yields

$$\eta\|f\|_2\frac{\sqrt{|\Lambda|}}{N} \leqslant C\sqrt{p}N^{\frac{1}{p}-1}\|f\|_{p'},$$

- Simplifying,

$$\sqrt{|\Lambda|} \leqslant C\eta^{-1}\sqrt{p}N^{\frac{1}{p}}\frac{\|f\|_{p'}}{\|f\|_2}.$$

- Taking $p = \log(N)$

$$|\Lambda| \leqslant C\eta^{-2}\log N\frac{\|f\|_{p'}^2}{\|f\|_2^2},$$

- thus proving (4).

- To prove (5) observe that

$$
\begin{aligned}
\|f\|_{L^{p'}(\mu)} &= N^{\frac{1}{p}-1}\|f\|_{p'} \\
&= N^{\frac{1}{p}-1}\left(\sum_{x\in\mathbb{Z}_N}|f(x)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&= N^{\frac{1}{p}-1}\left(\|f\|_1\sum_{x\in\mathbb{Z}_N}\frac{|f(x)|}{\|f\|_1}|f(x)|^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \\
&\leqslant N^{\frac{1}{p}-1}\left(\|f\|_1\left(\sum_{x\in\mathbb{Z}_N}\frac{|f(x)|}{\|f\|_1}|f(x)|\right)^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}
\end{aligned}
$$

- by Jensen's inequality.

- This implies

$$\|f\|_{L^{p'}(\mu)} \leqslant N^{\frac{1}{p}-1}\|f\|_1^{\frac{p-1}{p}} \left(\frac{\|f\|_2^2}{\|f\|_1}\right)^{\frac{1}{p}}$$
$$= N^{\frac{1}{p}-1}\|f\|_1 \left(\frac{\|f\|_2}{\|f\|_1}\right)^{\frac{2}{p}} .$$

- Again combining with our previous bounds,

$$\eta\|f\|_2\frac{\sqrt{|\Lambda|}}{N} \leqslant C\sqrt{p}N^{\frac{1}{p}-1}\|f\|_1 \left(\frac{\|f\|_2}{\|f\|_1}\right)^{\frac{2}{p}} .$$

- This simplifies to

$$\sqrt{|\Lambda|} \leqslant C\eta^{-1}\sqrt{p}\left(\frac{\|f\|_2^2}{\|f\|_1^2}N\right)^{\frac{1}{p}}\frac{\|f\|_1}{\|f\|_2}.$$

- Now, taking

$$p = \log\left(\frac{\|f\|_2^2}{\|f\|_1^2}N\right),$$

we obtain

$$|\Lambda| \leqslant C\eta^{-2}\log\left(\frac{\|f\|_2^2}{\|f\|_1^2}N\right)\frac{\|f\|_1}{\|f\|_2},$$

and this proves (5). □

- Numerical experiments show $FR(f)$ indicates structure, and also show the constants in Bourgain and Talagrand are small.
- We can approximate f by a trig polynomial of lower degree (L^2 and L^∞ approximation theorems)
- Looking at the large Fourier Coeffs for approx gives a deterministic low degree polynomial. (Sparse Spectrum Theorem)
- We hope to use the above approximation results to show that the collection

$$\mathcal{H}_M = \{f : \mathbb{Z}_N \to \mathbb{C} \mid FR(f) \leqslant M\}$$

has small statistical dimension.

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