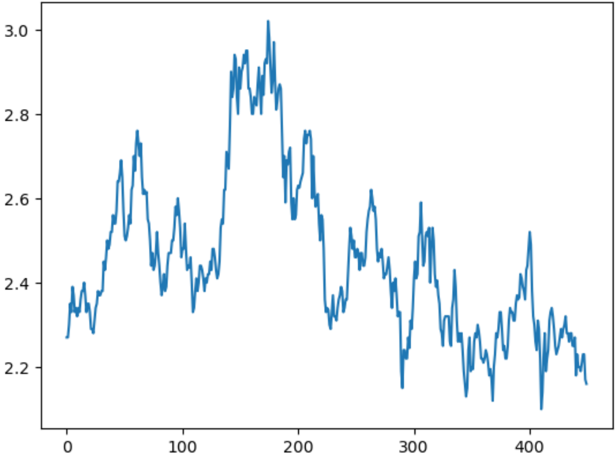
Fourier uncertainty and exact signal recovery

Alex Iosevich

January 2025: Colloquium talk at Northwestern

U.S. Inflation Rate



Classical imputation using trig polynomial regression

- In the picture below, 150 of the 450 points in the original inflation data set are randomly removed. The values are then filled in using the trig polynomial regression. What you see is the graph of the original missing points (red) and the imputed values (black).

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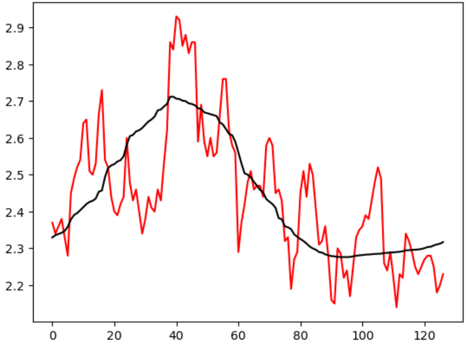

Imputation using the methods of exact signal recovery

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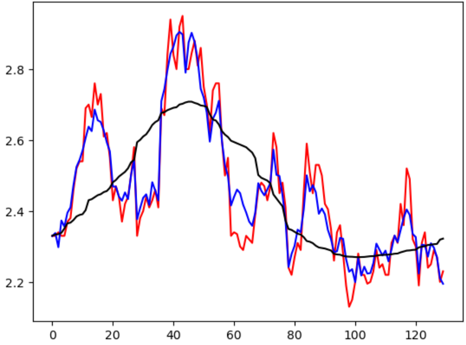

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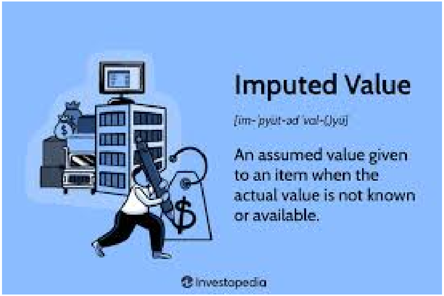

Finite Signals and Discrete Fourier transform

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- Suppose that the Fourier transform of f is transmitted, where

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- Fourier Inversion says that we can recover the signal by using the Fourier inversion:

$$f(x) = N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \widehat{f}(m).$$

Exact recovery problem

- The basic question is, can we recover f **exactly** from its discrete Fourier transforms if

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- The answer turns out to be *YES* if f is supported in $E \subset \mathbb{Z}_N^d$, and

$$|E| \cdot |S| < \frac{N^d}{2},$$

with the main tool being the Fourier Uncertainty Principle.

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- (Plancherel)

$$\sum_{m \in \mathbb{Z}_N^d} |\widehat{f}(m)|^2 = \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2.$$

A basic inequality

- Using the triangle inequality, it is not difficult to see that

$$|\widehat{f}(m)| \leq N^{-\frac{d}{2}} \sum_{x \in \mathbb{Z}_N^d} |\chi(-x \cdot m) f(x)|$$

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- Similarly,

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An elementary point of view: setup

- Suppose that $E \subset \mathbb{Z}_N^d$ and $f(x) = 1_E(x)$, the indicator function of E.

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- $$= I(x) + II(x).$$

An elementary point of view: direct estimation

- By the triangle inequality,

$$|II(x)| \leq N^{-\frac{d}{2}} \cdot |S| \cdot N^{-\frac{d}{2}} \cdot |E| = N^{-d} \cdot |E| \cdot |S|.$$

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- Since we know nothing about S, the best we can do is hope that the quantity above is small.

An elementary point of view: rounding

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$$N^{-d}|E||S| < \frac{1}{2},$$
we can take the modulus of $I(x)$ and round it up to 1 if it is $\geq \frac{1}{2}$, and round it down to 0 otherwise.

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- But what happens if we consider general signals?

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- If f cannot be recovered uniquely, then there exists a signal $g : \mathbb{Z}_N^d \to \mathbb{C}$ such that g also has $|E|$ non-zero entries,

$$\widehat{f}(m) = \widehat{g}(m) \text{ for } m \notin S,$$

and f is not identically equal to g.

Mysterious Property $\rightarrow$ Unique Recovery

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- It would then follow that if we assume

$$|E| \cdot |S| < \frac{N^d}{2},$$

we must have $h = 0$, and hence the recovery is *unique*.

A bit more notation

- Given $f : \mathbb{Z}_N^d \to \mathbb{C}$, define

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- We also define

$$||f||_{L^2(\mathbb{Z}_N^d)} = \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^2 \right)^{\frac{1}{2}}.$$

Mysterious Property is the Fourier Uncertainty Principle

- Let's prove that if $h : \mathbb{Z}_N^d \to \mathbb{C}$ such that $T = \{x : h(x) \neq 0\}$ and $S = \{m : \widehat{h}(m) \neq 0\}$, then

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- Summing both sides over $x \in T$ and cancelling the L^1 norms, we obtain

$$|T| \cdot |S| \geq N^d.$$

The method of least squares

- The uniqueness proof above also suggests an algorithm for recovering the missing information, albeit not a very efficient one. Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ with $\{\widehat{f}(m)\}_{m \in S}$ unobserved, and let

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- then f can be recovered as

$$arg\ min_u\ ||u||_{L^1(\mathbb{Z}_N^d)} \text{ subject to } \widehat{f}(m) = \widehat{u}(m) \text{ for } m \notin S.$$

Benjamin Franklin Logan

- Logan was an accomplished bluegrass musician in addition to his groundbreaking work in signal processing.

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Proof of the L^1 recovery method

- Let $f = g + h$, where g is the solution to the L^1 minimization problem above, and note that $\widehat{h}$ is supported in S. We have

$$||g||_{L^1(\mathbb{Z}_N^d)} = ||f - h||_{L^1(\mathbb{Z}_N^d)}$$

$$= ||f - h||_{L^1(E)} + ||h||_{L^1(E^c)} \geq ||f||_{L^1(\mathbb{Z}_N^d)} + \left[||h||_{L^1(E^c)} - ||h||_{L^1(E)} \right].$$

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- If we can show that $||h||_{L^1(E^c)} > ||h||_{L^1(E)}$, then

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which is impossible since g is the L^1 minimizer.

- The resulting contradiction will prove that $h \equiv 0$.

The uncertainty principle strikes again

- We have

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- It follows that

$$||h||_{L^1(E)} \leq N^{-d} \cdot |E| \cdot |S| \cdot ||h||_{L^1(\mathbb{Z}_N^d)} < \frac{1}{2} \cdot ||h||_{L^1(\mathbb{Z}_N^d)}.$$

We conclude that

$$||h||_{L^1(E)} < ||h||_{L^1(E^c)},$$

as desired.

Can we loosen the $|E| \cdot |S| < \frac{N^d}{2}$ assumption?

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- In general, the answer is no. Suppose that $d = 1$, N **is not** prime, and E is a subgroup of $\mathbb{Z}_N$.

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- However, we shall that for a generic set S of missing frequencies, the situation is much better.

The prime case $d \geq 2$

- If N is prime and $d \geq 2$, it is not difficult to check that if f is supported on a k-dimension plane H, $\widehat{f}$ is supported on the orthogonal subspace $H^{\perp}$.

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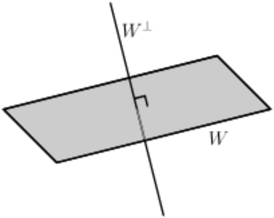

The prime case $d = 1$

- If N is prime and $d = 1$, a beautiful result due to Terry Tao says that if f is supported on E and $\widehat{f}$ is supported on S, then

$$|E| + |S| \geq N + 1,$$

with the corresponding improvement for the exact signal recovery condition.

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  with the corresponding improvement for the exact signal recovery condition.

- This result is a consequence of a beautiful 1924 result due to Chebotaryov, which says that every k by k minor of the Fourier matrix

$$\left\{ e^{-\frac{2\pi ixm}{N}} \right\}_{x \in \mathbb{Z}_N, m \in \mathbb{Z}_N}$$

  is non-singular if N is an odd prime.

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$$\left|\left|\sum_{i \in S} a_i \phi_i\right|\right|_{L^q(G)} \leq C(q) \cdot \left(\sum_{i \in S} |a_i|^2\right)^{\frac{1}{2}},$$

where $C(q)$ depends only on q.

Bourgain's Λ_q theorem - general formulation

- Jean Bourgain proved that if G is a locally compact abelian group, $\phi_1, \dots, \phi_n$ are orthogonal functions with $||\phi_j||_\infty \leq 1$, the for a generic set $S \subset \{1, 2, \dots, n\}$ of size $\approx n^{\frac{2}{q}}$, $q > 2$,

$$\left|\left|\sum_{i \in S} a_i \phi_i\right|\right|_{L^q(G)} \leq C(q) \cdot \left(\sum_{i \in S} |a_i|^2\right)^{\frac{1}{2}},$$

  where $C(q)$ depends only on q.

- As we shall see, this result has a simple and effective built-in uncertainty principle.

Jean Bourgain

- Jean Bourgain (Fields Medal 1994) was one of the greatest mathematicians who ever lived. Reading his papers is an incredibly rewarding, if somewhat painful, process.

The meaning of generic

- The notion of **generic** above means the following. Let $0 < \delta < 1$ and let $\{\xi_j\}_{1 \leq j \leq n}$ denote independent $0, 1$ random variables of mean $\int \xi_j(\omega) d\omega = \delta$, $1 \leq j \leq n$.

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- Choosing $\delta = n^{\frac{2}{q}-1}$ generates a random subset

$$S_\omega = \{1 \leq j \leq n : \xi_j(\omega) = 1\} \text{ of } \{1, 2, \dots n\}$$

of expected size $\lceil n^{\frac{2}{q}} \rceil$. Bourgain's theorem holding for a **generic** set S means that the result holds for the set S_ω with probability $1 - o_N(1)$.

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- In a simpler language, if we randomly choose a subset of $\{1, 2, \dots, n\}$ by choosing each element with probability $p = n^{\frac{2}{q}-1}$, then Bourgain's theorem holds for such a set with probability close to 1.

Bourgain's Λ_q theorem

- It is a consequence of Bourgain's celebrated Λ_p theorem in locally compact abelian groups that if $f : \mathbb{Z}_N^d \to \mathbb{C}$ and $\widehat{f}$ **is supported in** S, then for a "generic" set of size $\lceil N^{\frac{2d}{q}} \rceil$, $2 < q < \infty$,

$$\left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^q \right)^{\frac{1}{q}} \leq C(q) \left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2 \right)^{\frac{1}{2}},$$

with $C(q)$ independent of N.

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  with $C(q)$ independent of N.

- It is not difficult to see that this inequality implies that the support of f must be a positive proportion of $\mathbb{Z}_N^d$.

Signal recovery in the presence of the Λ_q inequality

Theorem

(A. Iosevich and A. Mayeli (2024)) Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ *be a signal supported in* $E \subset \mathbb{Z}_N^d$. *Suppose that the frequencies* $\{\widehat{f}(m)\}_{m \in S}$ *are unobserved, where* S *satisfies the* Λ_q *inequality with constant* $C(q)$, *i.e whenever* $\widehat{g}$ *is supported in* S, $|S| = \lceil N^{\frac{2d}{q}} \rceil$,

$$\left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |g(x)|^q \right)^{\frac{1}{q}} \leq C(q) \left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |g(x)|^2 \right)^{\frac{1}{2}},$$

with $C(q)$ *independent of* N. *Then* f *can be recovered exactly provided that*

$$|E| < \frac{N^d}{2(C(q))^{\frac{1}{\frac{1}{2} - \frac{1}{q}}}},$$

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A direct consequence of Bourgain's Λ_q theorem

- Suppose that S is generic, as in Bourgain's theorem.

- Suppose that f is supported in $E \subset \mathbb{Z}_N^d$ and $\widehat{f}$ is supported in S. Bourgain's theorem implies that

- $$N^{-\frac{d}{q}} \cdot |E|^{\frac{1}{q}} \left(\frac{1}{|E|} \sum_{x \in E} |f(x)|^q \right)^{\frac{1}{q}}$$
$$\leq C(q) N^{-\frac{d}{2}} \cdot |E|^{\frac{1}{2}} \left(\frac{1}{|E|} \sum_{x \in E} |f(x)|^2 \right)^{\frac{1}{2}}.$$

A direct consequence of Bourgain's Λ_q theorem

- It follows that
$$|E| \geq \frac{N^d}{(C(q))^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}}.$$

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- We conclude that if we send the Fourier transform of a signal f supported on a set of size

$$< \frac{N^d}{2(C(q))^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}},$$

and the frequencies in $S \subset \mathbb{Z}_N^d$, $|S| = \lceil N^{\frac{2d}{q}} \rceil$, satisfying the Λ_q, $q > 2$, inequality with constant $C(q)$ are missing, we can recover f exactly with very high probability using the rather inefficient L^2 method described above.

Talagrand's theorem (general)

- The general form of Talagrand's theorem (the first result of this type following Bourgain's 1989 result) is the following.

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Theorem

Let $\{\phi_j\}_{j=1}^n$ be an orthonormal system in L^2 with $|\phi_j(x)| \leq 1$, $1 \leq j \leq n$. There exists a constant $\gamma_0 \in (0,1)$ and a generic subset $I \subset \{1,\ldots,n\}$ with $|I| \geq \gamma_0 n$ such that for every $a = (a_i) \in \mathbb{C}^n$,

$$\left(\sum_{i \in I} |a_i|^2 \right)^{\frac{1}{2}} \leq C_T \sqrt{\log(n) \log\log(n)} \cdot \left\| \sum_{i \in I} a_i \phi_i \right\|_{L^1}.$$

where $C_T > 0$ is a universal constant.

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Talagrand's theorem in our context

- In the context of functions mapping $\mathbb{Z}_N^d \to \mathbb{C}$, Talagrand's theorem takes on the following form.

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Theorem

There exists $\gamma_0 \in (0,1)$ such that if $h : \mathbb{Z}_N^d \to \mathbb{C}$ with $\widehat{h}$ supported in a generic set S of size $|S| \geq \gamma_0 N^d$, then with probability $1 - o_N(1)$,

$$\left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |h(x)|^2 \right)^{\frac{1}{2}} \leq C_T \sqrt{\log(N^d) \log\log(N^d)} \cdot \frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |h(x)|.$$

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Michel Talagrand

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A slight variant of a result due to Iosevich, Kashin, Limonova, and Mayeli (2024)

Theorem

There exists $\gamma_0 \in (0,1)$ such that if $f : \mathbb{Z}_N^d \to \mathbb{C}$ with $\{\widehat{f}(m)\}_{m \in S}$ unobserved, where S is a generic set of size $\approx \gamma_0 N^d$, then if

$$|\{x \in \mathbb{Z}_N^d : f(x) \neq 0\}| \leq \frac{1}{4C_T} \frac{N^d}{\log(N^d) \log\log(N^d)},$$

then f can be recovered uniquely using the algorithm

$$g = argmin_u ||u||_1 \text{ with the constraint } \widehat{u}(m) = \widehat{f}(m) \text{ for } m \notin S.$$

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Back to the time series imputation

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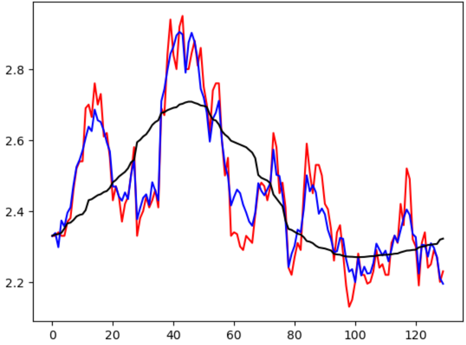

The key theorem behind the blue imputation engine

Theorem

(Burstein, Iosevich, Mayeli and Nathan) Let $f : \mathbb{Z}_N \to \mathbb{C}$, and suppose that the values $\{f(x)\}_{x \in M}$ are unobserved, where M is a generic subset of $\mathbb{Z}_N$. Let

$$g = argmin_u ||\widehat{u}||_1 \text{ with the constraint } u(x) = f(x) \text{ for } x \notin M.$$

Suppose that $\widehat{f}$ is ϵ-concentrated on $S \subset \mathbb{Z}_N$ in the sense that $||\widehat{f}||_{L^2(S^c)} \leq \frac{\epsilon}{N} \cdot ||\widehat{f}||_{L^2(\mathbb{Z}_N)}$, such that $|S| < \frac{1}{16C_T^2} \frac{N}{\log(N) \log\log(N)}$. Let $h = f - g$. Then if $h \neq 0$, then with probability $1 - o_N(1)$,

$$\frac{\frac{1}{|M|} \sum_{x \in M} |h(x)|}{\frac{1}{N} \sum_{x \in \mathbb{Z}_N} |f(x)|} \leq 4\epsilon.$$

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Eh...
thanks
a lot!