Logan's L^1-minimization idea and signal recovery

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Finite Signals and Discrete Fourier transform

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- Fourier Inversion says that we can recover the signal by writing

$$f(x) = N^{-\frac{1}{2}} \sum_{m \in \mathbb{Z}_N} \chi(x \cdot m) \widehat{f}(m).$$

Proof of Fourier Inversion

- We have

$$N^{-\frac{1}{2}} \sum_{m=0}^{N-1} e^{\frac{2\pi imx}{N}} \widehat{f}(m) = \sum_{m=0}^{N-1} e^{\frac{2\pi imx}{N}} \cdot N^{-\frac{1}{2}} \cdot \sum_{y=0}^{N-1} e^{\frac{-2\pi imy}{N}} f(y)$$

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- since

$$N^{-1} \sum_{m=0}^{N-1} e^{\frac{2\pi i m(x-y)}{N}}$$

is equal to 1 if $x = y$ and 0 otherwise, by a simple geometric series argument.

Exact recovery problem

- The basic question is, can we recover f **exactly** from its discrete Fourier transforms if

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- Let us first understand the obvious limitations. If all the values of $\widehat{f}$ are missing, we are lost. As it turns out, we can say a lot more.

A simple example

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- The expression above is equal to 0 unless m is a multiple of q, in which case it equals $\sqrt{\frac{q}{p}}$. In other words, $\widehat{1}_E(m) = \sqrt{\frac{q}{p}} \cdot 1_S(m)$, where S is the set of multiples of q in $\mathbb{Z}_{pq}$.

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- If the missing frequencies of $\widehat{1}_E(m)$ are in S, we cannot recover 1_E. Since $|E| \cdot |S| = pq = N$, this exhibits a natural limitation on a general recovery mechanism.

L^1 minimization algorithm

- Donoho and Stark (1989) showed, using a beautiful idea due to Benjamin Logan, that if $f : \mathbb{Z}_N \to \mathbb{C}$ is supported in E, and the frequencies $\{\widehat{f}(m)\}_{m \in S}$ are unobserved, then if

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- We shall unpack the definitions above in a moment.

Definition of arg min

- Let $F : X \to \mathbb{R}$ be a real-valued function on a set X. Then

$$\arg\min_{x \in X} F(x)$$

denotes the set of all points $x^\star \in X$ at which F attains its minimum value:

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- In the L^1 minimization problem from earlier, arg min selects the function u with the smallest L^1 norm that matches the known values of $\widehat{f}(m)$ for $m \notin S$.

Definition of $L^1(\mathbb{Z}_N)$

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$$L^1(\mathbb{Z}_N) = \left\{ f : \mathbb{Z}_N \to \mathbb{C} : \sum_{x \in \mathbb{Z}_N} |f(x)| < \infty \right\}.$$

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- Given $f : \mathbb{Z}_N \to \mathbb{C}$, we define

$$||f||_{L^1(\mathbb{Z}_N)} = \sum_{x \in \mathbb{Z}_N} |f(x)|.$$

Benjamin Franklin Logan (1927-2015)

- Logan was an accomplished bluegrass musician in addition to his groundbreaking scientific work . His most famous publication is the song, entitled "Christmas Time is Coming", performed by Johnny Cash and many others.

Proof of the L^1 recovery method

- Let $f = g + h$, where g is the solution to the L^1 minimization problem above, and note that $\widehat{h}$ is supported in S. We have

$$||g||_{L^1(\mathbb{Z}_N)} = ||f - h||_{L^1(\mathbb{Z}_N)}$$

$$= ||f - h||_{L^1(E)} + ||h||_{L^1(E^c)} \geq ||f||_{L^1(\mathbb{Z}_N)} + \left[||h||_{L^1(E^c)} - ||h||_{L^1(E)} \right].$$

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- If we can show that $||h||_{L^1(E^c)} > ||h||_{L^1(E)}$, then

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- The resulting contradiction will prove that $h \equiv 0$. Please note that this implies that the arg min gives a unique value in this case.

The triangle inequality

- We have

$$|h(x)| = N^{-\frac{1}{2}} \cdot \left| \sum_{m \in S} \chi(x \cdot m) \widehat{h}(m) \right| \leq N^{-1} \cdot |S| \cdot ||h||_{L^1(\mathbb{Z}_N)}.$$

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- It follows that

$$||h||_{L^1(E)} \leq N^{-1} \cdot |E| \cdot |S| \cdot ||h||_{L^1(\mathbb{Z}_N)} < \frac{1}{2} \cdot ||h||_{L^1(\mathbb{Z}_N)}.$$

We conclude that

$$||h||_{L^1(E)} < ||h||_{L^1(E^c)},$$

as desired.

Example: Sparse Signal with Composite $N = 8$

- Let $f : \mathbb{Z}_8 \to \mathbb{R}$ be defined by:

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ -1 & \text{if } x = 4 \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad \text{support size } |E| = 2$$

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- The discrete Fourier transform is equal to

$$\widehat{f}(m) = \frac{1}{\sqrt{8}}\left(e^{-2\pi i m/8} - e^{-2\pi i 4m/8}\right) = \frac{1}{\sqrt{8}}\left(e^{-2\pi i m/8} - (-1)^m\right)$$

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- Then

$$|\widehat{f}(m)| = \frac{2}{\sqrt{8}} \cdot \left| \sin\left(\frac{\pi m}{2}\right) \right| = \frac{1}{\sqrt{2}} \cdot \left| \sin\left(\frac{\pi m}{2}\right) \right|$$

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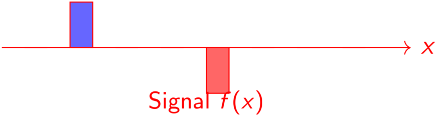
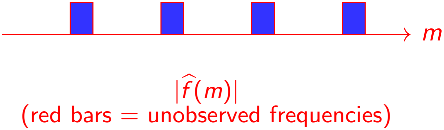
- Suppose frequencies $m = 2$ and $m = 6$ are unobserved. Then $|S| = 2$, and since $|E| \cdot |S| = 4 < 8 = N$, recovery via L^1 minimization is possible.

Diagram: Sparse Signal and Its Fourier Magnitudes

- Visualizing $f(x)$ and $|\widehat{f}(m)|$ with $m = 2, 6$ unobserved (red bars):






Eh...
thanks
a lot!