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Subsystems of orthogonal systems and the recovery of sparse signals in the presence of random losses

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In what follows by signals we mean vectors $f=\{f_i\}_{i=1}^N$ in $\mathbb{C}^N$ with the standard scalar product $\langle\,\cdot\,,\cdot\,\rangle$, and we set $\operatorname{supp} f=\{i\colon f_i\neq 0\}$. Also let $\langle N\rangle=\{1,2,\dots,N\}$, and let $|A|$ be the cardinality of the finite set A. If $\Phi=\{\varphi_j\}_{j=1}^N$ is a fixed orthonormal basis in $\mathbb{C}^N$ and $f\in\mathbb{C}^N$, then we let $\widehat{f}_j$, $j\in\langle N\rangle$, denote the Fourier coefficients of the signal f: $\widehat{f}_j=\langle f,\varphi_j\rangle$.

In this note we continue the investigations of the problem considered in [1], on the recovery of a sparse signal f from information about the magnitude of just some Fourier coefficients $\{\widehat{f}_j, j\in\langle N\rangle\setminus\Lambda\}$ in the case when the set Λ of lost coefficients is of stochastic nature. For details of the motivations behind this problem, see [1] and [2].

Theorem 1 below, as well as Corollaries 4.1 and 4.2 in [1], are based on deep facts in the theory of general orthogonal series. While in [1] this was Bourgain's classical theorem on $\Lambda(p)$-sets, here we use Theorem 3 in [3] and some estimates from [3] established in the proof of that theorem.

Let an orthonormal basis $\Phi=\{\varphi_j\}_{j=1}^N$ in $\mathbb{C}^N$ be fixed so that

$$\|\varphi_j\|_{l_\infty^N}\leqslant\frac{K}{N^{1/2}}\,,\qquad j\in\langle N\rangle. \tag{1}$$

Given $k\in\langle N-1\rangle$, on a probability space (Γ,P) we consider a system of independent vector-valued variables $\{X_\nu\}_{\nu=1}^k$ such that for $j\in\langle N\rangle$ X_ν takes the value φ_j with probability $1/N$. Each point $\gamma\in\Gamma$ gives rise to a random system of functions $\{X_\nu(\gamma)\}_{\nu=1}^k$ and a random subsystem of Φ: $\{\varphi_j, j\in\Lambda_\gamma^k\}\equiv\Phi_\gamma^k\equiv\Phi\setminus\{X_\nu(\gamma)\}_{\nu=1}^k$.

For each k, $1\leqslant k<N$, consider the set $G(k)\subset\Gamma$ of points γ such that for each polynomial Q with respect to Φ_γ^k we have

$$\|Q\|_{l_2^N}\leqslant\frac{K}{[R(N)]^{1/2}}\,\|Q\|_{l_1^N},\qquad R(N)\equiv\frac{N}{\log(10N)(\log\log(10N))^6}\,. \tag{2}$$

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It follows from the proof of Proposition 1 and Theorem 3 in [3] that for each K there exist sequences $\{k_N\}_{N=1}^{\infty}$, $\lim\limits_{N\to\infty} k_N/N = 0$, and $\{\rho_N\}_{N=1}^{\infty}$, $\lim\limits_{N\to\infty} \rho_N = 0$, such that for each orthonormal basis Φ satisfying (1) we have

$$\mathsf{P}\bigl(G(k_N)\bigr) \geqslant 1 - \rho_N, \qquad N = 1, 2, \dots . \tag{3}$$

Note that for each polynomial $Q \not\equiv 0$ with respect to the system Φ_γ^k that has properties (1) and (2) we have $|\operatorname{supp} Q| \geqslant R(N)/K^2$. It follows directly from this relation that in this case two signals f and g such that $|\operatorname{supp} f| + |\operatorname{supp} g| < R(N)/K^2$ and $\widehat{f}_j = \widehat{g}_j$ for $j \in \langle N\rangle \setminus \Lambda_\gamma^k$ coincide identically.

Definition. We call a set $E \subset \langle N\rangle$ a stable recovery spectrum with parameter S for the orthonormal basis Φ if each signal $f \in \mathbb{C}^N$ such that $|\operatorname{supp} f| < S$ can uniquely be recovered from the values $\widehat{f}_j$, $j \in E$, using the l_1-minimization algorithm:

$$f = \arg\min\{\|g\|_{l_1^N} : \widehat{g}_j = \widehat{f}_j,\ j \in E\}.$$

We denote the set of stable recovery spectra by $\mathrm{SR}(\Phi, S)$.

Using (3) and repeating the arguments in [4] (see Lemmas 2.2 and 2.3 there) almost word-for-word, we can establish the following result.

Theorem 1. *For each real $K \geqslant 1$ there exist sequences $\{k_N\}_{N=1}^{\infty}$, $\lim\limits_{N\to\infty} k_N/N = 0$, and $\{\rho_N\}_{N=1}^{\infty}$, $\lim\limits_{N\to\infty} \rho_N = 0$, such that for $N = 1, 2, \dots$ the following probability estimate holds for each orthonormal basis Φ in $\mathbb{C}^N$ satisfying (1):*

$$\mathsf{P}\biggl\{\gamma \in \Gamma \colon \langle N\rangle \setminus \Lambda_\gamma^{k_N} \in \mathrm{SR}\biggl(\Phi, \frac{R(N)}{4K^2}\biggr)\biggr\} \geqslant 1 - \rho_N.$$

Remark 1. As shown, in fact, in [4], conditions of type (2) for a subsystem Φ_γ^k and $(N-k)/N$ close to 1 allow us not only to recover sparse signals precisely, but also to recover approximately signals f close to sparse ones from just a small share of their Fourier coefficients $\widehat{f}_j$, $j \in \langle N\rangle \setminus \Lambda_\gamma^k$.

Remark 2. In place of Bourgain's theorem used in [1], in the problem of the recovery of sparse signals we can use results from [5] and [6] on lacunary subsystems of orthonormal systems. However, results obtained in this way do not allow us to obtain stable recovery spectra of size $\leqslant N/2$.

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