Erdos's square root distance set argument

Alex Iosevich

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- What is the smallest possible size $\Delta(P)$ if the size of P is equal to n?

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- Does the example above yield the set of size n in $\mathbb{R}^2$ with the smallest possible number of distances? The answer turns out to be no, and we will get back to this subject matter later.

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- But what are we looking for and what if N is smaller than that number?

Circles centered at a point p

- Each circle represents all points at a fixed distance from p. The total number of such circles gives a lower bound on the number of distinct distances from p.

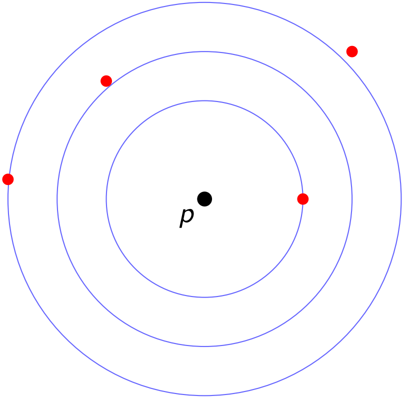

Figure: Circles centered at p, each containing one or more points from P.

Running into a wall and reacting properly

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- Choose the "right-most" point on **good** half of the circle and draw line segments to the other points on the **good** half of the circle, moving from right to left.

A rich circle and a bisecting line

- If N is small, some circle must contain many points—this is a “rich” circle.

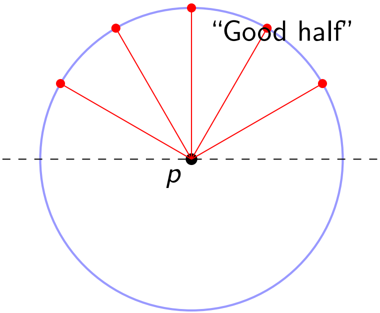

Figure: Dividing a rich circle into halves. The top half contains many points.

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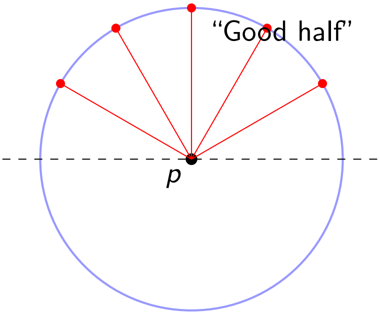
- A line through p can divide it into two halves, with one half still containing many points.



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- Moving right along, choose arbitrary points $(\cos(\theta_1), \sin(\theta_1))$ and $(\cos(\theta_2), \sin(\theta_2))$, where $0 < \theta_i \leq \pi$.

A geometric calculation

- The distance squared from $(1,0)$ to $(\cos(\theta_1),\sin(\theta_1))$ is equal to

$$D_1^2 = (1-\cos(\theta_1))^2 + \sin^2(\theta_1) = 2 - 2\cos(\theta_1),$$

while the distance squared from $(1,0)$ to $(\cos(\theta_2),\sin(\theta_2))$ is equal to

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- We want to show that $D_1^2 < D_2^2$, which amounts to showing that

$$\cos(\theta_2) < \cos(\theta_1) \text{ if } \theta_1 < \theta_2,$$

which is true because the cosine function is decreasing on $[0,\pi]$.

Distances on a half-circle increase

- The distances from $(1, 0)$ to points on the upper half-circle increase from right to left.

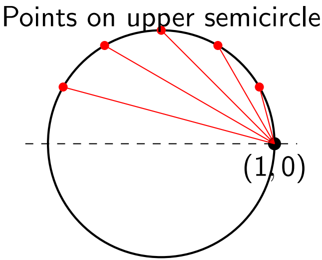

Figure: Distances from the fixed point $(1, 0)$ increase as angle θ increases.

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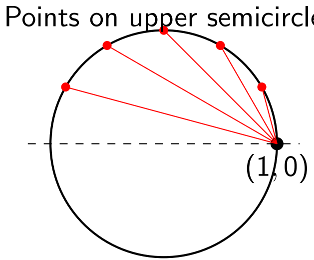
- This is due to cosine decreasing on $[0, \pi]$.



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Drawing conclusions

- We have just seen that if N is much smaller than n, then

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- It follows that

$$(\#\Delta(P))^2 \geq \frac{n-1}{2}, \text{ hence } \#\Delta(P) \geq \sqrt{\frac{n-1}{2}}.$$


Eh...
thanks
a lot!